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# Lecture Notes in Microeconomic Theory

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## The Economic Agent

*Ariel Rubinstein*

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# Preface

This short book contains my lecture notes for the first quarter of a microeconomics course for PhD or Master's degree economics students. The lecture notes were developed over a period of almost 15 years during which I taught the course, or parts of it, at Tel Aviv, Princeton, and New York universities.

I am publishing the lecture notes with some hesitation. Several superb books are already on the shelves. I most admire Kreps (1990), which pioneered the transformation of the game theoretic revolution in economic theory from research papers into textbooks. His book covers the material in depth and includes many ideas for future research. Mas-Colell, Whinston, and Green (1995) continued this trend with a very comprehensive and detailed textbook. There are three other books on my short list: Bowles (2003), which brings economics back to its authentic, political economics roots; Jehle and Reny (1997), with its very precise style; and the classic Varian (1984). These five books constitute an impressive collection of textbooks for the standard advanced microeconomics course.

My book covers only the first quarter of the standard course. It does not aim to compete but to supplement these books. I had it published only because I think that some of the didactic ideas in the book might be beneficial to students and teachers, and it is to this end that I insisted on retaining the lecture notes style.

Throughout the book I use only male pronouns. This is my deliberate choice and does not reflect the policy of the editors or the publishers. I believe that continuous reminders of the he/she issue simply divert readers' attention. Language is of course very important in shaping our thinking and I don't dispute the importance of the type of language we use. But I feel it is more effective to raise the issue of discrimination against women in the discussion of gender-related issues, rather than raising flags on every page of a book on economic theory.

A special feature of this book is that it is also posted on the Internet and access is entirely free. My intention is to update the book annually (or at least in years when I teach the course). To access the latest electronic version of the book, visit: http://arielrubinstein.tau.ac.il/micro1/.

I would like to thank all my teaching assistants, who contributed comments during the many years I taught the course: Rani Spiegler, Kfir Eliaz, Yoram Hamo, Gabi Gayer and Tamir Tshuva at Tel Aviv University; Bilge Yilmiz, Ronny Razin, Wojciech Olszewski, Attila Ambrus, Andrea Wilson, Haluk Ergin and Daisuke Nakajima at Princeton; and Sophie Bade and Anna Ingster at NYU. Special thanks are due to Sharon Simmer and Rafi Aviav who helped me with the English editing and to Gabi Gayer and Daniel Wasserteil who prepared the figures.

# Introduction

As a new graduate student, you are at the beginning of a new stage of your life. In a few months you will be overloaded with definitions, concepts, and models. Your teachers will be guiding you into the wonders of economics and will rarely have the time to stop to raise fundamental questions about what these models are supposed to mean. It is not unlikely that you will be brainwashed by the professional-sounding language and hidden assumptions. I am afraid I am about to initiate you into this inevitable process. Still, I want to use this opportunity to pause for a moment and alert you to the fact that many economists have strong and conflicting views about what economic theory is. Some see it as a *set of theories* that can (or should) be tested. Others see it as a *bag of tools* to be used by economic agents, and yet others see it as a *framework* through which professional and academic economists view the world.

My own view may disappoint those of you who have come to this course with practical motivations. In my view, economic theory is no more than an arena for the *investigation of concepts* we use in thinking about economics in real life. What makes a theoretical model "economics" is that the concepts we are analyzing are taken from real-life reasoning about economic issues. Through the investigation of these concepts we indeed try to understand reality better, and the models provide a language that enables us to think about economic interactions in a systematic way. But I do not view economic models as an attempt to describe the world or to provide tools for predicting the future. I object to looking for an ultimate truth in economic theory, and I do not expect it to be the foundation for any policy recommendation. Nothing is "holy" in economic theory and everything is the creation of people like yourself.

Basically, this course is about a certain class of economic *concepts* and *models*. Although we will be studying formal concepts and models, they will always be given an interpretation. An economic model differs substantially from a purely mathematical model in that it is a *combination* of a mathematical model and its *interpretation*. The names of the mathematical objects are an *integral* part of an economic model. When mathematicians use terms such as "field" or "ring" which are in everyday use, it is only for the sake of convenience. When they name a collection of sets a "filter," they are doing so in an associative manner; in principle, they could call it "ice cream cone." When they use the term "good ordering" they are not making an ethical judgment. In contrast to mathematics, interpretation is an essential ingredient of any economic model.

It is my hope that some of you will react and attempt to change what is currently called economic theory, and that some of you will acquire alternative ways of thinking about economic and social interactions. At the very least, the course should teach you to ask hard questions about economic models and in what sense they are relevant to the economic questions we are interested in. I hope that you walk away from this course with the recognition that the answers are not as obvious as they might appear.

## Microeconomics

In this course we deal only with microeconomics, a collection of models in which the primitives are details about the behavior of units called *economic agents*. Microeconomic models investigate assumptions about economic agents' activities and about interactions between these agents. An economic agent is the basic unit operating in the model. Most often, we do have in mind that the economic agent is an individual, a person with one head, one heart, two eyes, and two ears. However, in some economic models, an economic agent is taken to be a nation, a family, or a parliament. At other times, the "individual" is broken down into a collection of economic agents, each operating in distinct circumstances and each regarded as an economic agent.

We should not be too cheerful about the statement that an economic agent in microeconomics is not constrained to being an individual. The facade of generality in economic theory might be misleading. We have to be careful and aware that when we take an economic agent to be a group of individuals, the reasonable assumptions we might impose on it are distinct from those we might want to impose on a single individual. In any case, with a particular economic scenario in mind, the decision about how to think of that scenario in the framework of a microeconomic model involves a decision about whom *we* want to view as the primitives.

An economic agent is described in our models as a unit that responds to a scenario called a *choice problem*, where the agent must make a choice from a set of available alternatives. The economic agent appears in the microeconomic model with a specified deliberation process he uses to make a decision. In most of current economic theory, the deliberation process is what is called *rational* choice. The agent decides what action to take through a process in which he

1. asks himself "What is desirable?"
2. asks himself "What is feasible?"
3. chooses the most desirable from among the feasible alternatives.

Rationality in economics does not contain judgments about desires. A rational agent can have preferences which the entire world views as being against the agent's interest. Furthermore, economists are fully aware that almost all people, almost all the time, do not practice this kind of deliberation.

Nevertheless, we find the investigation of economic agents who follow the rational process to be important, since we often refer to rational decision making in life as an ideal process. It is meaningful to talk about the concept of "being good" even in a society where all people are evil; similarly, it is meaningful to talk about the concept of a "rational man" and about the interactions between rational economic agents even if all people systematically behave in a nonrational manner.

# Lecture Notes in Microeconomic Theory

LECTURE 1

# Preferences

## Preferences

Although we are on our way to constructing a model of rational choice, we begin the course with an "exercise": formulating the notion of "preferences" independently of the concept of choice. We view preferences as the mental attitude of an individual (economic agent) toward alternatives.We seek to develop a "proper" formalization of this concept, which plays such a central role in economics.

Imagine that you want to fully describe the preferences of an agent toward the elements in a given set $X$. For example, imagine that you want to describe your own attitude toward the universities you apply to before finding out to which of them you have been admitted. What must the description include? What conditions must the description fulfill?

We take the approach that a description of preferences should fully specify the attitude of the agent toward each pair of elements in $X$. For each pair of alternatives, it should provide an answer to the question of how the agent compares the two alternatives. We present two versions of this question. For each version we formulate the consistency requirements necessary to make the responses "preferences" and examine the connection between the two formalizations.

## The Questionnaire $Q$

Let us think about the preferences on a set $X$ as *answers* to a "long" questionnaire $Q$ which consists of all quiz questions of the type:

***Q(x,y)*** (for all distinct $x$ and $y$ in $X$)

How do you compare $x$ and $y$? Tick one and only one of the following three options:

☐ I prefer $x$ to $y$ (this answer is denoted as $x \succ y$).
☐ I prefer $y$ to $x$ (this answer is denoted by $y \succ x$).
☐ I am indifferent (this answer is denoted by $I$).

A "legal" answer to the questionnaire is a response in which the respondent ticks exactly one of the boxes in each question. We do not allow the decision maker to refrain from answering a question or to tick more than one answer. Furthermore, we do not allow him to respond with answers that demonstrate a lack of ability to compare, such as:

☐ They are incomparable.
☐ I don't know what $x$ is.
☐ I have no opinion.

Or a dependence on other factors such as:

☐ It depends on what my parents think.
☐ It depends on the circumstances (sometimes I prefer $x$ but usually I prefer $y$).

Or the intensity of preferences such as:

☐ I somewhat prefer $x$.
☐ I love $x$ and I hate $y$.

Or confusion such as:

☐ I both prefer $x$ over $y$ and $y$ over $x$.
☐ I can't concentrate right now.

The constraints that we place on the legal responses of the agents constitute our implicit assumptions. Particularly important are the assumptions that the elements in the set $X$ are all comparable, that the individual has an opinion about all elements in the set $X$ and that we do not allow him to specify the intensity of preferences.

A legal answer to the questionnaire can be formulated as a function $f$ which assigns to any pair $(x, y)$ of distinct elements in $X$ exactly one of the three "values": $x \succ y$ or $y \succ x$ or $I$, with the interpretation that $f(x, y)$ is the answer to the question $Q(x, y)$. (Alternatively, we can use the terminology of the soccer "betting" industry and say that $f(x, y)$ must be 1, 2, or × with the interpretation that

$f(x, y) = 1$ means that $x$ is better than $y$, $f(x, y) = 2$ means that $y$ is better than $x$ and $f(x, y) = \times$ means indifference.)

Not all legal answers to the questionnaire $Q$ qualify as *preferences over the set* $X$. We will adopt two "consistency" restrictions:

First, the answer to $Q(x, y)$ must be identical to the answer to $Q(y, x)$. In other words, we want to exclude the common "framing effect" by which people who are asked to compare two alternatives tend to prefer the "first" one.

Second, we require that the answers exhibit "transitivity." In other words, the answers to $Q(x, y)$ and $Q(y, z)$ must be consistent with the answer to $Q(x, z)$ in the following sense: If "$x$ is preferred to $y$" and "$y$ is preferred to $z$" then "$x$ is preferred to $z$," and if the answers to the two questions $Q(x, y)$ and $Q(y, z)$ are "indifference" then so is the answer to $Q(x, z)$.

To summarize, here is my favorite formalization of the notion of preferences:

### *Definition 1*

*Preferences on a set* $X$ are a function $f$ that assigns to any pair $(x, y)$ of distinct elements in $X$ exactly one of the three "values" $x \succ y$, $y \succ x$ or $I$ so that for any three different elements $x$, $y$ and $z$ in $X$, the following two properties hold:

- *No order effect*: $f(x, y) = f(y, x)$.
- *Transitivity*:
  if $f(x, y) = x \succ y$ and $f(y, z) = y \succ z$ then $f(x, z) = x \succ z$ and
  if $f(x, y) = I$ and $f(y, z) = I$ then $f(x, z) = I$.

Note again that $I$, $x \succ y$, and $y \succ x$ are merely symbols representing verbal answers. Needless to say, the choice of symbols is not an arbitrary one. (Why do I use the notation $I$ and not $x \sim y$?)

## A Discussion of Transitivity

The transitivity property is an appealing property of preferences. How would you react if somebody told you he prefers $x$ to $y$, $y$ to $z$ and $z$ to $x$? You would probably feel that his answers are "confused." Furthermore, it seems that, when confronted with an intransitivity in their responses, people are embarrassed and want to change their answers.

Before the lecture, students in Tel Aviv had to fill out a questionnaire similar to $Q$ regarding a set $X$ that contains nine alternatives, each specifying the following four characteristics of a travel package: location (Paris or Rome), price, quality of the food, and quality of the lodgings. The questionnaire included only thirty six questions since for each pair of alternatives $x$ and $y$, only one of the questions, $Q(x, y)$ or $Q(y, x)$, was randomly selected to appear in the questionnaire (thus the dependence on order of an individual's response could not be checked within the experimental framework). In the 2004 group, out of eighteen MA students, only two had no intransitivities in their answers, and the average number of triples in which intransitivity existed was almost nine. Many of the violations of transitivity involved two alternatives that were actually the same, but differed in the order in which the characteristics appeared in the description. "A weekend in Paris at a four-star hotel with food quality Zagat 17 for \$574," and "A weekend in Paris for \$574 with food quality Zagat 17 at a four-star hotel." All students expressed indifference between the two alternatives, but in a comparison of these two alternatives to a third alternative—"A weekend in Rome at a five-star hotel with food quality Zagat 18 for \$612"—half of the students gave responses that violated transitivity.

In spite of the appeal of the transitivity requirement, note that when we assume that the attitude of an individual toward pairs of alternatives is transitive, we are excluding individuals who base their judgments on "procedures" that cause systematic violations of transitivity. The following are two such examples.

1. *Aggregation of considerations as a source of intransitivity*. In some cases, an individual's attitude is derived from the aggregation of more basic considerations. Consider, for example, a case where $X = \{a, b, c\}$ and the individual has three primitive considerations in mind. The individual finds one alternative better than the other if a majority of considerations support the first alternative. This aggregation process can yield intransitivities. For example, if the three considerations rank the alternatives as follows: $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$ and $c \succ_3 a \succ_3 b$, then the individual determines $a$ to be preferred over $b$, $b$ over $c$, and $c$ over $a$, thus violating transitivity.

2. *The use of similarities as an obstacle to transitivity*. In some cases, the decision maker expresses indifference in a comparison between two elements that are too "close" to be distinguishable. For example, let $X = \Re$ (the set of real numbers). Consider an individual whose attitude is "the more the better"; however, he finds it impossible to determine whether $a$ is greater than $b$ unless the difference is at least 1. He will assign $f(x, y) = x \succ y$ if $x \geq y - 1$ and $f(x, y) = I$ if $|x - y| < 1$. This is not a preference relation since $1.5 \sim 0.8$ and $0.8 \sim 0.3$, but it is not true that $1.5 \sim 0.3$.

*Did we require too little?* Another potential criticism of our definition is that our assumptions might have been too weak and that we did not impose some reasonable further restrictions on the concept of preferences. That is, there are other similar consistency requirements we may impose on a legal response to qualify it as a description of preferences. For example, if $f(x, y) = x \succ y$ and $f(y, z) = I$, we would naturally expect that $f(x, z) = x \succ z$. However, this additional consistency condition was not included in the above definition since it follows from the other conditions: If $f(x, z) = I$, then by the assumption that $f(y, z) = I$ and by the no order effect, $f(z, y) = I$, and thus by transitivity $f(x, y) = I$ (a contradiction). Alternatively, if $f(x, z) = z \succ x$, then by no order effect $f(z, x) = z \succ x$, and by $f(x, y) = x \succ y$ and transitivity $f(z, y) = z \succ y$ (a contradiction).

Similarly, note that for any preferences $f$, we have if $f(x, y) = I$ and $f(y, z) = y \succ z$, then $f(x, z) = x \succ z$.

## The Questionnaire *R*

A second way to think about preferences is through an imaginary questionnaire $R$ consisting of all questions of the type:

***R(x,y)*** (for all $x, y \in X$, not necessarily distinct).

"Is $x$ at least as preferred as $y$?" Tick one and only one of the following two options:

☐ Yes
☐ No

By a "legal" response we mean that the respondent ticks exactly one of the boxes in each question. To qualify as preferences a legal response must also satisfy two conditions:

1. The answer to at least one of the questions $R(x, y)$ and $R(y, x)$ must be Yes. (In particular, the "silly" question $R(x, x)$ which appears in the questionnaire must get a Yes response.)
2. For every $x, y, z \in X$, if the answers to the questions $R(x, y)$ and $R(y, z)$ are Yes, then so is the answer to the question $R(x, z)$.

We identify a response to this questionnaire with the binary relation $\succsim$ on the set $X$ defined by $x \succsim y$ if the answer to the question $R(x, y)$ is Yes.

(*Reminder*: An *n-ary relation* on $X$ is a subset of $X^n$. Examples: "Being a parent of" is a binary relation on the set of human beings; "being a hat" is an unary relation on the set of objects; "$x + y = z$" is a 3-nary relation on the set of numbers; "$x$ is better than $y$ more than $x'$ is better than $y'$" is 4-nary relation on a set of alternatives, etc. An *n-ary relation* on $X$ can be thought of as a response to a questionnaire regarding all $n$-tuples of elements of $X$ where each question can get only a Yes/No answer.)

This brings us to the "traditional" definition:

***Definition 2***

A *preference on* a set $X$ is a binary relation $\succsim$ on $X$ satisfying:

- *Completeness*: For any $x, y \in X$, $x \succsim y$ or $y \succsim x$.
- *Transitivity*: For any $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

**The Equivalence of the Two Definitions**

We have presented two definitions of preferences on the set $X$. We now proceed to show their equivalence. There are many ways to construct "a one-to-one correspondence" between the objects satisfying the two definitions. But, when we think about the equivalence of two definitions in economics we are thinking about much more than the existence of a one-to-one correspondence: the correspondence has to *preserve the interpretation*. Note the similarity to the notion of an isomorphism in mathematics. For example, an isomorphism between two topological spaces $X$ and $Y$ is a one-to-one

**Table 1.1**

| A response to $Q(x, y)$ and $Q(y, x)$ | A response to $R(x, y)$ and $R(y, x)$ |
|---|---|
| $x \succ y$ | Yes, No |
| $I$ | Yes, Yes |
| $y \succ x$ | No, Yes |

function from $X$ onto $Y$ that is required to preserve the open sets. In economics, the one-to-one correspondence is required to preserve the more informal concept of interpretation.

We will now construct a one-to-one and onto correspondence, *Translation*, between answers to $Q$ that qualify as preferences by the first definition and answers to $R$ that qualify as preferences by the second definition, such that the correspondence preserves the meaning of the responses to the two questionnaires. In other words, *Translation* is a "bridge" between the responses to $Q$ that qualify as preferences and the responses to $R$ that qualify as preferences.

To illustrate the correspondence imagine that you have two books. Each page in the first book is a response to the questionnaire $Q$ which qualifies as preferences by the first definition. Each page in the second book is a response to the questionnaire $R$ which qualifies as preferences by the second definition. The correspondence matches each page in the first book with a unique page in the second book, so that a reasonable person will recognize that the different responses to the two questionnaires reflect the same mental attitudes towards the alternatives.

Since we assume that the answers to all questions of the type $R(x, x)$ are "Yes," the classification of a response to $R$ as a preference only requires the specification of the answers to questions $R(x, y)$, where $x \neq y$. Table 1.1 presents the translation of responses.

This translation preserves the interpretation we have given to the responses, that is, "I prefer $x$ to $y$" has the same meaning as the statement "I find $x$ to be at least as good as $y$, but I don't find $y$ to be at least as good as $x$."

The following observations complete the proof that *Translation* is indeed a one-to-one correspondence from the set of preferences, as given by definition 1, onto the set of preferences as given by definition 2.

By the assumption on $Q$ of a no order effect, for any two alternatives $x$ and $y$, one and only one of the following three answers was received for both $Q(x,y)$ and $Q(y,x)$: $x \succ y$, $I$ and $y \succ x$. Thus, the responses to $R(x,y)$ and $R(y,x)$ are well defined.

Next we verify that the response to $R$ that we have constructed with the table is indeed a preference relation (by the second definition).

Completeness: In each of the three rows, the answers to at least one of the questions $R(x,y)$ and $R(y,x)$ is affirmative.

Transitivity: Assume that the answers to $R(x,y)$ and $R(y,z)$ are affirmative. This implies that the answer to $Q(x,y)$ is either $x \succ y$ or $I$, and the answer to $Q(y,z)$ is either $y \succ z$ or $I$. Transitivity of $Q$ implies that the answer to $Q(x,z)$ must be $x \succ z$ or $I$, and therefore the answer to $R(x,z)$ must be affirmative.

To see that *Translation* is indeed a one-to-one correspondence, note that for any two different responses to the questionnaire $Q$ there must be a question $Q(x,y)$ for which the responses differ; therefore, the corresponding responses to either $R(x,y)$ or $R(y,x)$ must differ.

It remains to be shown that the range of the *Translation* function includes all possible preferences as defined by the second definition. Let $\succsim$ be preferences in the traditional sense (a response to $R$). We have to specify a function $f$, a response to $Q$, which is converted by *Translation* to $\succsim$. Read from right to left, the table provides us with such a function $f$.

By the completeness of $\succsim$, for any two elements $x$ and $y$, one of the entries in the right-hand column is applicable (the fourth option, that the two answers to $R(x,y)$ and $R(y,x)$ are "No," is excluded), and thus the response to $Q$ is well defined and by definition satisfies no order effect.

We still have to check that $f$ satisfies the transitivity condition. If $F(x,y) = x \succ y$ and $F(y,z) = y \succ z$, then $x \succsim y$ and not $y \succsim x$ and $y \succsim z$ and not $z \succsim y$. By transitivity of $\succsim$, $x \succsim z$. In addition, not $z \succsim x$ since if $z \succsim x$, then the transitivity of $\succsim$ would imply $z \succsim y$. If $F(x,y) = I$ and $F(y,z) = I$, then $x \succsim y$, $y \succsim x$, $y \succsim z$ and $z \succsim y$. By transitivity of $\succsim$, both $x \succsim z$ and $z \succsim x$, and thus $F(x,z) = I$.

### Summary

From now on we will use the second definition, that is, a *preference on $X$* is a binary relation $\succsim$ on a set $X$ satisfying Completeness and

Transitivity. For a preference relation $\succsim$, we will use the notation $x \sim y$ when both $x \succsim y$ and $y \succsim x$; the notation $x \succ y$ will stand for if $x \succsim y$ and not $y \succsim x$.

## Bibliographic Notes

*Recommended readings*: Kreps 1990, 17–24; Mas-Colell et al. 1995, chapter 1, A–B.

Fishburn (1970) contains a comprehensive treatment of preference relations.

# Problem Set 1

**Problem 1.** (*Easy*)
Let $\succsim$ be a preference relation on a set X. Define $I(x)$ to be the set of all $y \in X$ for which $y \sim x$.

Show that the set (of sets!) $\{I(x) | x \in X\}$ is a partition of $X$, i.e.,

- For all $x$ and $y$, either $I(x) = I(y)$ or $I(x) \cap I(y) = \emptyset$.
- For every $x \in X$, there is $y \in X$ such that $x \in I(y)$.

**Problem 2.** (*Standard*)
Kreps (1990) introduces another formal definition for preferences. His primitive is a binary relation $P$ interpreted as "strictly preferred." He requires $P$ to satisfy:

- *Asymmetry*: For no $x$ and $y$ do we have both $xPy$ and $yPx$.
- *Negative-Transitivity*: For all $x$, $y$, and $z \in X$, if $xPy$, then for any $z$ either $xPz$ or $zPy$ (or both).

Explain the sense in which Kreps' formalization is equivalent to the traditional definition.

**Problem 3.** (*Standard*)
In economic theory we are often interested in other types of binary relations, for example, the relation $xSy$: "$x$ and $y$ are almost the same." Suggest properties that would correspond to your intuition about such a concept.

**Problem 4.** (*Difficult.* Based on Kannai and Peleg 1984.)
Let $Z$ be a finite set and let $X$ be the set of all nonempty subsets of $Z$. Let $\succsim$ be a preference relation on $X$ (not $Z$).

Consider the following two properties of preference relations on $X$:

a. If $A \succsim B$ and $C$ is a set disjoint to both $A$ and $B$, then $A \cup C \succsim B \cup C$, and
   if $A \succ B$ and $C$ is a set disjoint to both $A$ and $B$, then $A \cup C \succ B \cup C$.
b. If $x \in Z$ and $\{x\} \succ \{y\}$ for all $y \in A$, then $A \cup \{x\} \succ A$, and
   if $x \in Z$ and $\{y\} \succ \{x\}$ for all $y \in A$, then $A \succ A \cup \{x\}$.

Discuss the plausibility of the properties in the context of interpreting $\succsim$ as the attitude of the individual toward sets from which he will have to make a choice at a "second stage."

Provide an example of a preference relation that

- Satisfies the two properties.
- Satisfies the first but not the second property.
- Satisfies the second but not the first property.

Show that if there are $x, y$, and $z \in Z$ such that $\{x\} \succ \{y\} \succ \{z\}$, then there is no preferene relation satisfying both properties.

**Problem 5.** (*Fun*)
Listen to the illusion called the Shepard Scale. (Currently, it is available at http://www.sandlotscience.com/Ambiguous/ShpTones1.htm and http://asa.aip.org/demo27.html.)

Can you think of any economic analogies?

LECTURE 2

# Utility

## The Concept of Utility Representation

Think of examples of preferences. In the case of a small number of alternatives, we often describe a preference relation as a list arranged from best to worst. In some cases, the alternatives are grouped into a small number of categories and we describe the preferences on $X$ by specifying the preferences on the set of categories. But, in my experience, most of the examples that come to mind are similar to: "I prefer the taller basketball player," "I prefer the more expensive present," "I prefer a teacher who gives higher grades," "I prefer the person who weighs less."

Common to all these examples is that they can naturally be specified by a statement of the form "$x \succsim y$ if $V(x) \geq V(y)$" (or $V(x) \leq V(y)$), where $V : X \to \Re$ is a function that attaches a real number to each element in the set of alternatives $X$. For example, the preferences stated by "I prefer the taller basketball player" can be expressed formally by: $X$ is the set of all conceivable basketball players, and $V(x)$ is the height of player $x$.

Note that the statement $x \succsim y$ if $V(x) \geq V(y)$ always defines a preference relation since the relation $\geq$ on $\Re$ satisfies completeness and transitivity.

Even when the description of a preference relation does not involve a numerical evaluation, we are interested in an equivalent numerical representation. We say that *the function* $U : X \to \Re$ *represents the preference* $\succsim$ if for all $x$ and $y \in X$, $x \succsim y$ if and only if $U(x) \geq U(y)$. If the function $U$ represents the preference relation $\succsim$, we refer to it as a *utility function* and we say that $\succsim$ has a *utility representation*.

It is possible to avoid the notion of a utility representation and to "do economics" with the notion of preferences. Nevertheless, we usually use utility functions rather than preferences as a means of describing an economic agent's attitude toward alternatives, probably because we find it more convenient to talk about the maximization of a numerical function than of a preference relation.

Note that when defining a preference relation using a utility function, the function has an intuitive meaning that carries with it additional information. In contrast, when the utility function is formed in order to represent an existing preference relation, the utility function has no meaning other than that of representing a preference relation. Absolute numbers are meaningless in the latter case; only relative order has meaning. Indeed, if a preference relation has a utility representation, then it has an infinite number of such representations, as the following simple claim shows:

***Claim:***

If $U$ represents $\succsim$, then for any strictly increasing function $f : \Re \to \Re$, the function $V(x) = f(U(x))$ represents $\succsim$ as well.

***Proof:***

$a \succsim b$
iff $U(a) \geq U(b)$ (since $U$ represents $\succsim$)
iff $f(U(a)) \geq f(U(b))$ (since $f$ is strictly increasing)
iff $V(a) \geq V(b)$.

**Existence of a Utility Representation**

If any preference relation could be represented by a utility function, then it would "grant a license" to use utility functions rather than preference relations with no loss of generality. Utility theory investigates the possibility of using a numerical function to represent a preference relation and the possibility of numerical representations carrying additional meanings (such as, $a$ is preferred to $b$ more than $c$ is preferred to $d$).

We will now examine the basic question of "utility theory": Under what assumptions do utility representations exist?

Our first observation is quite trivial. When the set $X$ is finite, there is always a utility representation. The detailed proof is presented here mainly to get into the habit of analytical precision. We start with a lemma regarding the existence of minimal elements (an element $a \in X$ is *minimal* if $a \precsim x$ for any $x \in X$).

***Lemma:***

In any finite set $A \subseteq X$ there is a minimal element (similarly, there is also a maximal element).

***Proof:***

By induction on the size of $A$. If $A$ is a singleton, then by completeness its only element is minimal.

For the inductive step, let $A$ be of cardinality $n+1$ and let $x \in A$. The set $A-\{x\}$ is of cardinality $n$ and by the inductive assumption has a minimal element denoted by $y$. If $x \succsim y$, then $y$ is minimal in $A$. If $y \succsim x$, then by transitivity $z \succsim x$ for all $z \in A-\{x\}$ and thus $x$ is minimal.

***Claim:***

If $\succsim$ is a preference relation on a finite set $X$, then $\succsim$ has a utility representation with values being natural numbers.

***Proof:***

We will construct a sequence of sets inductively. Let $X_1$ be the subset of elements that are minimal in $X$. By the above lemma, $X_1$ is not empty. Assume we have constructed the sets $X_1, \ldots, X_k$. If $X = X_1 \cup X_2 \cup \ldots \cup X_k$ we are done. If not, define $X_{k+1}$ to be the set of minimal elements in $X - X_1 - X_2 - \cdots - X_k$. By the lemma $X_{k+1} \neq \emptyset$. Since $X$ is finite we must be done after at most $|X|$ steps. Define $U(x) = k$ if $x \in X_k$. Thus, $U(x)$ is the step number at which $x$ is "eliminated." To verify that $U$ represents $\succsim$, let $a \succsim b$. Then $b \notin X - X_1 - X_2 - \cdots - X_{U(a)}$ and thus $U(a) \geq U(b)$.

Without any further assumptions on the preferences, the existence of a utility representation is guaranteed when the set $X$ is countable (recall that $X$ is countable and infinite if there is a one-to-one function from the natural numbers to $X$, namely, it is possible to specify an enumeration of all its members $\{x_n\}_{n=1,2,\ldots}$).

***Claim:***

If $X$ is countable, then any preference relation on $X$ has a utility representation with a range $(-1, 1)$.

***Proof:***

Let $\{x_n\}$ be an enumeration of all elements in $X$. We will construct the utility function inductively. Set $U(x_1) = 0$. Assume that you have completed the definition of the values $U(x_1), \ldots, U(x_{n-1})$ so that $x_k \succsim x_l$ iff $U(x_k) \geq U(x_l)$. If $x_n$ is indifferent to $x_k$ for some $k < n$, then assign $U(x_n) = U(x_k)$. If not, by transitivity, all numbers in the set $\{U(x_k) | \; x_k \prec x_n\} \cup \{-1\}$ are below all numbers in the set $\{U(x_k) | \; x_n \prec x_k\} \cup \{1\}$. Choose $U(x_n)$ to be between the two sets. This guarantees that for any $k < n$ we have $x_n \succsim x_k$ iff $U(x_n) \geq U(x_k)$. Thus, the function we defined on $\{x_1, \ldots, x_n\}$ represents the preference on those elements.

To complete the proof that $U$ represents $\succsim$, take any two elements, $x$ and $y \in X$. For some $k$ and $l$ we have $x = x_k$ and $y = x_l$. The above applied to $n = \max\{k, l\}$ yields $x_k \succsim x_l$ iff $U(x_k) \geq U(x_l)$.

**Lexicographic Preferences**

Lexicographic preferences are the outcome of applying the following procedure for determining the ranking of any two elements in a set $X$. The individual has in mind a sequence of criteria that could be used to compare pairs of elements in $X$. The criteria are applied in a fixed order until a criterion is reached that succeeds in distinguishing between the two elements, in that it determines the preferred alternative. Formally, let $(\succsim_k)_{k=1,\ldots,K}$ be a $K$-tuple of orderings over the set $X$. The lexicographic ordering induced by those orderings is defined by $x \succsim_L y$ if (1) there is $k^*$ such that for all $k < k^*$ we have $x \sim_k y$ and $x \succ_{k*} y$ or (2) $x \sim_k y$ for all $k$. Verify that $\succsim_L$ is a preference relation.

***Example:***

Let $X$ be the unit square, i.e., $X = [0, 1] \times [0, 1]$. Let $x \succsim_k y$ if $x_k \geq y_k$. The lexicographic ordering $\succsim_L$ induced from $\succsim_1$ and $\succsim_2$ is: $(a_1, a_2) \succsim_L (b_1, b_2)$ if $a_1 > b_1$ or both $a_1 = b_1$ and $a_2 \geq b_2$. (Thus, in this example, the left component is the primary criterion while the right component is the secondary criterion.)

We will now show that the preferences $\succsim_L$ do *not* have a utility representation. The lack of a utility representation excludes lexicographic preferences from the scope of standard economic models in spite of the fact that they constitute a simple and commonly used procedure for preference formation.

***Claim:***

The preference relation $\succsim_L$ on $[0, 1] \times [0, 1]$, which is induced from the relations $x \succsim_k y$ if $x_k \geq y_k$ $(k = 1, 2)$, does not have a utility representation.

***Proof:***

Assume by contradiction that the function $u : X \to \Re$ represents $\succsim_L$. For any $a \in [0, 1]$, $(a, 1) \succ_L (a, 0)$ we thus have $u(a, 1) > u(a, 0)$. Let $q(a)$ be a rational number in the nonempty interval $I_a = (u(a, 0), u(a, 1))$. The function $q$ is a function from $X$ into the set of rational numbers. It is a one-to-one function since if $b > a$ then $(b, 0) \succ_L (a, 1)$ and therefore $u(b, 0) > u(a, 1)$. It follows that the intervals $I_a$ and $I_b$ are disjoint and thus $q(a) \neq q(b)$. But the cardinality of the rational numbers is lower than that of the continuum, a contradiction.

## Continuity of Preferences

In economics we often take the set $X$ to be an infinite subset of a Euclidean space. The following is a condition that will guarantee the existence of a utility representation in such a case. The basic intuition, captured by the notion of a continuous preference relation,

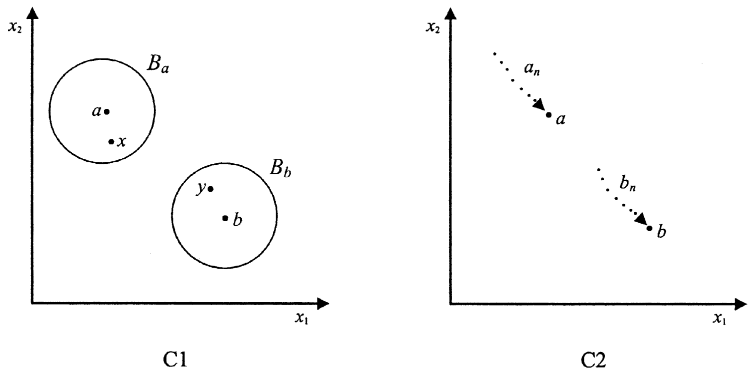



**Figure 2.1**
Two definitions of continuity of preferences.

is that if $a$ is preferred to $b$, then "small" deviations from $a$ or from $b$ will not reverse the ordering.

***Definition C1:***

A preference relation $\succsim$ on $X$ is *continuous* if whenever $a \succ b$ (namely, it is not true that $b \succsim a$), there are neighborhoods (balls) $B_a$ and $B_b$ around $a$ and $b$, respectively, such that for all $x \in B_a$ and $y \in B_b$, $x \succ y$ (namely, it is not true that $y \succsim x$). (See fig. 2.1.)

***Definition C2:***

A preference relation $\succsim$ on $X$ is *continuous* if the graph of $\succsim$ (that is, the set $\{(x, y)|x \succsim y\} \subseteq X \times X$) is a closed set (with the product topology); that is, if $\{(a_n, b_n)\}$ is a sequence of pairs of elements in $X$ satisfying $a_n \succsim b_n$ for all $n$ and $a_n \to a$ and $b_n \to b$, then $a \succsim b$. (See fig. 2.1.)

***Claim:***

The preference relation $\succsim$ on $X$ satisfies C1 if and only if it satisfies C2.

***Proof:***

Assume that $\succsim$ on $X$ is continuous according to C1. Let $\{(a_n, b_n)\}$ be a sequence of pairs satisfying $a_n \succsim b_n$ for all $n$ and $a_n \to a$ and $b_n \to b$. If it is not true that $a \succsim b$ (that is, $b \succ a$), then there exist two balls $B_a$ and $B_b$ around $a$ and $b$, respectively, such that for all $y \in B_b$ and $x \in B_a$, $y \succ x$. There is an $N$ large enough such that for all $n > N$, both $b_n \in B_b$ and $a_n \in B_a$. Therefore, for all $n > N$, we have $b_n \succ a_n$, which is a contradiction.

Assume that $\succsim$ is continuous according to C2. Let $a \succ b$. Denote by $B(x, r)$ the set of all elements in $X$ distanced less than $r$ from $x$. Assume by contradiction that for all $n$ there exist $a_n \in B(a, 1/n)$ and $b_n \in B(b, 1/n)$ such that $b_n \succsim a_n$. The sequence $(b_n, a_n)$ converges to $(b, a)$; by the second definition $(b, a)$ is within the graph of $\succsim$, that is, $b \succsim a$, which is a contradiction.

***Remarks***

1. If $\succsim$ on $X$ is represented by a *continuous* function $U$, then $\succsim$ is continuous. To see this, note that if $a \succ b$ then $U(a) > U(b)$. Let $\varepsilon = (U(a) - U(b))/2$. By the continuity of $U$, there is a $\delta > 0$ such that for all $x$ distanced less than $\delta$ from $a$, $U(x) > U(a) - \varepsilon$, and for all $y$ distanced less than $\delta$ from $b$, $U(y) < U(b) + \varepsilon$. Thus, for $x$ and $y$ within the balls of radius $\delta$ around $a$ and $b$, respectively, $x \succ y$.

2. The lexicographic preferences which were used in the counterexample to the existence of a utility representation are not continuous. This is because $(1, 1) \succ (1, 0)$, but in any ball around $(1, 1)$ there are points inferior to $(1, 0)$.

3. Note that the second definition of continuity can be applied to any binary relation over a topological space, not just to a preference relation. For example, the relation $=$ on the real numbers $(\Re^1)$ is continuous while the relation $\neq$ is not.

**Debreu's Theorem**

Debreu's theorem, which states that continuous preferences have a *continuous* utility representation, is one of the classic results in economic theory. For a complete proof of Debreu's theorem see Debreu 1954, 1960. Here we prove only that continuity guarantees the existence of a utility representation.

***Lemma:***

If $\succsim$ is a continuous preference relation on a convex set $X \subseteq \Re^n$, and if $x \succ y$, then there exists $z$ in $X$ such that $x \succ z \succ y$.

***Proof:***

Assume not. Construct a sequence of points on the interval that connects the points $x$ and $y$ in the following way. First define $x_0 = x$ and $y_0 = y$. In the inductive step we have two points, $x_t$ and $y_t$, on the line that connects $x$ and $y$, such that $x_t \succsim x$ and $y \succsim y_t$. Consider the middle point between $x_t$ and $y_t$ and denote it by $m$. According to the assumption, either $m \succsim x$ or $y \succsim m$. In the former case define $x_{t+1} = m$ and $y_{t+1} = y_t$, and in the latter case define $x_{t+1} = x_t$ and $y_{t+1} = m$. The sequences $\{x_t\}$ and $\{y_t\}$ are converging, and they must converge to the same point $z$ since the distance between $x_t$ and $y_t$ converges to zero. By the continuity of $\succsim$ we have $z \succsim x$ and $y \succsim z$ and thus, by transitivity, $y \succsim x$, contradicting the assumption that $x \succ y$.

***Comment on the Proof:***

Another proof could be given for the more general case, in which the assumption that the set $X$ is convex is replaced by the assumption that it is a connected subset of $\Re^n$. Remember that a connected set cannot be covered by two disjoint open sets. If there is no $z$ such that $x \succ z \succ y$, then $X$ is the union of two disjoint sets $\{a|a \succ y\}$ and $\{a|x \succ a\}$, which are open by the continuity of the preference relation.

Recall that a set $Y \subseteq X$ is *dense* in $X$ if in every open subset of $X$ there is an element in $Y$. For example, the set $Y = \{x \in \Re^n |\ x_k$ is a rational number for $k = 1, .., n\}$ is a countable dense set in $\Re^n$.

***Proposition:***

Assume that $X$ is a convex subset of $\Re^n$ that has a countable dense subset $Y$. If $\succsim$ is a continuous preference relation, then $\succsim$ has a (continuous) utility representation.

***Proof:***

By a previous claim we know that there exists a function $v : Y \to [-1, 1]$, which is a utility representation of the preference relation $\succsim$ restricted to $Y$. For every $x \in X$, define $U(x) = sup\{v(z)|z \in Y \text{ and } x \succ z\}$. Define $U(x) = -1$ if there is no $z \in Y$ such that $x \succ z$, which means that $x$ is the minimal element in $X$. (Note that for $z \in Y$ it could be that $U(z) < v(z)$.)

If $x \sim y$, then $x \succ z$ iff $y \succ z$. Thus, the sets on which the supremum is taken are the same and $U(x) = U(y)$.

If $x \succ y$, then by the lemma there exists $z$ in $X$ such that $x \succ z \succ y$. By the continuity of the preferences $\succsim$ there is a ball around $z$ such that all the elements in that ball are inferior to $x$ and superior to $y$. Since $Y$ is dense, there exists $z_1 \in Y$ such that $x \succ z_1 \succ y$. Similarly, there exists $z_2 \in Y$ such that $z_1 \succ z_2 \succ y$. Finally,

$U(x) \geq v(z_1)$ (by the definition of $U$ and $x \succ z_1$),
$v(z_1) > v(z_2)$ (since $v$ represents $\succsim$ on $Y$ and $z_1 \succ z_2$), and
$v(z_2) \geq U(y)$ (by the definition of $U$ and $z_2 \succ y$).

## Bibliographic Notes

*Recommended readings*: Kreps 1990, 30–32; Mas-Colell et al. 1995, chapter 3, C.

Fishburn (1970) covers the material in this lecture very well. The example of lexicographic preferences originated in Debreu (1959) (see also Debreu 1960, in particular Chapter 2, which is available online at http://cowles.econ.yale.edu/P/cp/p00b/p0097.pdf.)

# Problem Set 2

**Problem 1.** (*Easy*)
The purpose of this problem is to make sure that you fully understand the basic concepts of utility representation and continuous preferences.

a. Is the statement "if both $U$ and $V$ represent $\succsim$ then there is a *strictly* monotonic function $f : \Re \to \Re$ such that $V(x) = f(U(x))$" correct?
b. Can a continuous preference be represented by a discontinuous function?
c. Show that in the case of $X = \Re$, the preference relation that is represented by the discontinuous utility function $u(x) = [x]$ (the largest integer $n$ such that $x \geq n$) is not a continuous relation.
d. Show that the two definitions of a continuous preference relation (C1 and C2) are equivalent to

***Definition C3:*** For any $x \in X$, the upper and lower contours $\{y |\ y \succsim x\}$ and $\{y |\ x \succsim y\}$ are closed sets in $X$,

and to

***Definition C4:*** For any $x \in X$, the sets $\{y |\ y \succ x\}$ and $\{y |\ x \succ y\}$ are open sets in $X$.

**Problem 2.** (*Moderate*)
Give an example of preferences over a countable set in which the preferences cannot be represented by a utility function that returns only integers as values.

**Problem 3.** (*Moderate*)
Consider the sequence of preference relations $(\succsim^n)_{n=1,2,..}$, defined on $\Re^2_+$ where $\succsim^n$ is represented by the utility function $u_n(x_1, x_2) = x_1^n + x_2^n$. We will say that the sequence $\succsim^n$ converges to the preferences $\succsim^*$ if for every $x$ and $y$, such that $x \succ^* y$, there is an $N$ such that for every $n > N$ we have $x \succ^n y$. Show that the sequence of preference relations $\succsim^n$ converges to the preferences $\succ^*$ which are represented by the function $max\{x_1, x_2\}$.

**Problem 4.** (*Moderate*)
The following is a typical example of a utility representation theorem:
Let $X = \Re^2_+$. Assume that a preference relation $\succsim$ satisfies the following three properties:
*ADD*: $(a_1, a_2) \succsim (b_1, b_2)$ implies that $(a_1 + t, a_2 + s) \succsim (b_1 + t, b_2 + s)$ for all $t$ and $s$.
*MON*: If $a_1 \geq b_1$ and $a_2 \geq b_2$, then $(a_1, a_2) \succsim (b_1, b_2)$; in addition, if either $a_1 > b_1$ or $a_2 > b_2$, then $(a_1, a_2) \succ (b_1, b_2)$.
*CON*: Continuity.

a. Show that if $\succsim$ has a linear representation (that is, $\succsim$ are represented by a utility function $u(x_1, x_2) = \alpha x_1 + \beta x_2$ with $\alpha > 0$ and $\beta > 0$), then $\succsim$ satisfies *ADD*, *MON* and *CON*.
b. Suggest circumstances in which ADD makes sense.
c. Show that the three properties are necessary for $\succsim$ to have a linear representation. Namely, show that for any pair of the three properties there is a preference relation that does not satisfy the third property.
d. (*This part is difficult*) Show that if $\succsim$ satisfies the three properties, then it has a linear representation.

**Problem 5.** (*Moderate*)
Utility is a numerical representation of preferences. One can think about the numerical representation of other abstract concepts. Here, you will try to come up with a possible numerical representation of the concept "approximately the same" (see Luce (1956) and Rubinstein (1988)). For simplicity, let $X$ be the interval $[0, 1]$.

Consider the following six properties of $S$:

(S-1) For any $a \in X$, $aSa$.
(S-2) For all $a, b \in X$, if $aSb$ then $bSa$.
(S-3) Continuity (the graph of the relation $S$ in $X \times X$ is a closed set).
(S-4) Betweenness: If $d \geq c \geq b \geq a$ and $dSa$ then also $cSb$.
(S-5) For any $a \in X$ there is an interval around $a$ such that $xSa$ for every $x$ in the interval.
(S-6) Denote $M(a) = max\{x|xSa\}$ and $m(a) = min\{x|aSx\}$. Then, $M$ and $m$ are (weakly) increasing functions and are strictly increasing whenever they do not have the values 0 or 1.

a. Do these assumptions capture your intuition about the concept "approximately the same"?
b. Show that the relation $S_\varepsilon$, defined by $aS_\varepsilon b$ if $|b - a| \leq \varepsilon$ (for positive $\varepsilon$), satisfies all assumptions.

c. (*Difficult*) Let $S$ be a binary relation that satisfies the above six properties and let $\varepsilon$ be a strictly positive number. Show that there is a strictly increasing and continuous function $H : X \to \Re$ such that $aSb$ if and only if $|H(a) - H(b)| \leq \varepsilon$ .

**Problem 6.** (*Reading*)
Read Kahneman (2000) (it is available at http://arielrubinstein.tau.ac.il/econt/k.pdf) and discuss his distinction between the different types of "psychological utilities."

LECTURE 3

# Choice

## Choice Functions

Until now we have avoided any reference to behavior. We have talked about preferences as a summary of the decision maker's mental attitude toward a set of alternatives. But economics is about behavior, and therefore we now move on to modeling an agent's choice. The term "agent's behavior" contains not only the specification of the agent's actual choices made when he confronts certain choice problems, it also contains a full description of his behavior in all scenarios we imagine he might confront.

Consider a *grand set* $X$ of possible alternatives. We view a choice problem as a nonempty subset of $X$, and we refer to a choice from $A \subseteq X$ as specifying one of $A$'s members. We think about behavior as a hypothetical response to a questionnaire that contains many questions of the following type:

**Q(A)**: Assume you have to choose from a set of alternatives $A$. Which alternative would you choose?____

A legal response to this questionnaire requires responding to *all* questions by indicating *a unique* element in $A$ for every question $Q(A)$.

In some contexts, not all questions are meaningful. Therefore we allow that the questionnaire consist of a subset of questions, one for each element of a set $D$ of subsets of $X$. We will refer to a pair $(X, D)$ as a *context*.

### *Example:*

Imagine that we are interested in a student's behavior regarding his selection from the set of universities to which he has been admitted.

Let $X = \{x_1, \ldots, x_N\}$ be the set of all universities in the scope of the student's imagination. A choice problem $A$ is interpreted as the set of universities to which he has been admitted. The fact that the student was admitted to some subset of universities does not imply his admission outcome for other universities. Therefore, $D$ contains the $2^N - 1$ nonempty subsets of $X$. But if, for example, the universities are listed according to difficulty in being admitted ($x_1$ being the most difficult) and if the fact that the student is admitted to $x_k$ means that he is admitted to all less prestigious universities, that is, to all $x_l$ with $l > k$, then $D$ will consist of the $N$ sets $A_1, \ldots, A_N$ where $A_k = \{x_k, \ldots, x_N\}$.

Given a context $(X, D)$, a *choice function* $C$ assigns to each set $A \in D$ a unique element of $A$ with the interpretation that $C(A)$ is the chosen element from the set $A$.

Our understanding is that a decision maker behaving in accordance with the function $C$ will choose $C(A)$ if he has to make a choice from a set $A$. This does not mean that we can actually observe the choice function. At most we might observe some particular choices made by the decision maker in some instances. Thus, a choice function is a description of hypothetical behavior.

### Rational Choice Functions

It is typically assumed in economics that choice is an outcome of "rational deliberation". Namely, the decision maker has in mind a preference relation $\succsim$ on the set $X$ and, given any choice problem $A$ in $D$, he chooses an element in $A$ which is $\succsim$ optimal. Assuming that it is well defined, we define the *induced choice function* $C_{\succsim}$ as the function that assigns to every nonempty set $A \in D$ the $\succsim$-best element of $A$. Note that the preference relation is fixed, that is, it is independent of the choice set being considered.

### Dutch Book Arguments

Some of the justifications for this assumption are normative, that is, they reflect a perception that people should be rational in this sense and, if they are not, they should convert to reasoning of this type. One interesting class of arguments that aimed at supporting this approach is referred to in the literature as "Dutch book arguments."

The claim is that an economic agent who behaves according to a choice function that is not induced from maximization of a preference relation will not survive.

The following is a "sad" story about a monkey in a forest with three trees, $a$, $b$, and $c$. The monkey is about to pick a tree to sleep in. It has in mind a binary relation $\succsim$ that reflects the comparison he makes mentally between any two trees such that $a \succ b$, $b \succ c$, and $c \succ a$. Assume that whenever he is on tree $a$ he sees only tree $b$, whenever he is on tree $b$ he sees only tree $c$, and whenever he is on tree $c$ he observes only tree $a$. The monkey's choice function is $C(\{a,b\}) = b$, $C(\{b,c\}) = c$, $C(\{a,c\}) = a$. The monkey will perpetually jump from tree to tree to tree—not a good mode of behavior in the "cruel" environment of nature.

A similar "story," more appropriate to human beings, is called the "money pump" argument. Assume that a decision maker behaves like the monkey regarding three alternatives $a$, $b$, and $c$. Assume that (for all $x$ and $y$) the choice $C(x,y) = y$ is strong enough that while he is "holding" the option to receive the alternative $x$, he is ready to pay 1¢ for the ability to make the choice from $\{x,y\}$. In this case, he can be "pumped" for his money by giving him $a$ and offering him to replace what he holds with $b$, $c$, and again $a$ until his pockets are emptied, or until the decision maker learns his lesson and changes his behavior.

I bring this "Dutch book argument" here not as a necessarily convincing argument for rationality but just as an interesting argument. The above argument could be easily criticized. Its appeal requires, in particular, that we be convinced that the environment in which the economic agent operates would offer the agent the above sequence of choice problems.

## Rationalizing

Economists were often criticized for making the assumption that decision makers maximize a preference relation. The most common response to this criticism is that we don't really need this assumption. All we need to assume is that the decision maker's behavior can be described *as if* he were maximizing some preference relation.

Let us state this "economic defense" more precisely. We will say that a choice function $C$ *can be rationalized* if there is a preference relation $\succsim$ on $X$ so that $C = C_{\succsim}$ (that is, $C(A) = C_{\succsim}(A)$ for any $A$ in the domain of $C$).

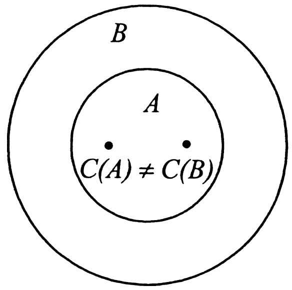



**Figure 3.1**
Violation of condition $*$.

We will now identify a condition under which a choice function can indeed be presented as if derived from some preference relation (i.e., can be rationalized).

***Condition $*$:***

We say that $C$ satisfies condition $*$ if for any two problems $A,B \in D$, if $A \subset B$ and $C(B) \in A$ then $C(A) = C(B)$. (See fig. 3.1.)

Note that if $\succsim$ is a preference relation on $X$, then $C_{\succsim}$ (defined on a set of subsets of $X$ that have a single most preferred element) satisfies $*$.

Alternatively, consider the "second-best procedure" in which the decision maker has in mind an ordering $\succsim$ of $X$ and for any given choice problem set $A$ chooses the element from $A$, which is the $\succsim$-maximal from the nonoptimal alternatives. The second-best procedure does not satisfy condition $*$: If $A$ contains all the elements in $B$ besides the $\succsim$-maximal, then $C(B) \in A \subset B$ but $C(A) \neq C(B)$.

We will now show that condition $*$ is a sufficient condition for a choice function to be formulated *as if* the decision maker is maximizing some preference relation.

***Proposition:***

Assume that $C$ is a choice function with a domain containing at least all subsets of $X$ of size no greater than 3. If $C$ satisfies $*$, then there is a preference $\succsim$ on $X$ so that $C = C_{\succsim}$.

***Proof:***

Define $\succsim$ by $x \succsim y$ if $x = C(\{x, y\})$.

Let us first verify that the relation $\succsim$ is a preference relation.

- *Completeness*: Follows from the fact that $C(\{x, y\})$ is always well defined.
- *Transitivity*: If $x \succsim y$ and $y \succsim z$, then $C(\{x, y\}) = x$ and $C(\{y, z\}) = y$. If $C(\{x, z\}) \neq x$ then $C(\{x, z\}) = z$. By $*$ and $C(\{x, z\}) = z$ , $C(\{x, y, z\}) \neq x$. By $*$ and $C(\{x, y\}) = x$, $C(\{x, y, z\}) \neq y$, and by $*$ and $C(\{y, z\}) = y$, $C(\{x, y, z\}) \neq z$. A contradiction to $C(\{x, y, z\}) \in \{x, y, z\}$.

We still have to show that $C(B) = C_{\succsim}(B)$. Assume that $C(B) = x$ and $C_{\succsim}(B) \neq x$. That is, there is $y \in B$ so that $y \succ x$ . By definition of $\succsim$, this means $C(\{x, y\}) = y$, contradicting $*$.

## What Is an Alternative

Some of the cases where rationality is violated can be attributed to the incorrect specification of the space of alternatives. Consider the following example taken from Luce and Raiffa (1957): A diner in a restaurant chooses *chicken* from the menu {*steak tartare*, *chicken*} but chooses *steak tartare* from the menu {*steak tartare*, *chicken*, *frog legs*}. At first glance it seems that he is not "rational" (since his choice conflicts with $*$). Assume that the motivation for the choice is that the existence of *frog legs* is an indication of the quality of the chef. If the dish *frog legs* is on the menu, the cook must then be a real expert, and the decision maker is happy ordering *steak tartare*, which requires expertise to make. If the menu lacks *frog legs*, the decision maker does not want to take the risk of choosing *steak tartare*.

Rationality is "restored" if we make the distinction between "*steak tartare* served in a restaurant where *frog legs* are also on the menu (and the cook must then be a real chef)" and "*steak tartare* in a restaurant where *frog legs* are not served (and the cook is likely a novice)." Such a distinction makes sense since the *steak tartare* is not the same in the two choice sets.

The lesson from the above discussion is that we should be careful in specifying the term "alternative." Note, however, that defining any alternative in terms of its physical description and the choice set from which it is to be chosen would empty the rationality hypothesis of its meaning.

## Choice Functions as "Internal Equilibria"

The choice function definition we have been using requires that a single element be assigned to each choice problem. If the decision maker follows the rational-man procedure using a preference relation with indifferences, the previously defined induced choice function $C_{\succsim}(A)$ might be undefined because for some choice problems there would be more than one optimal element. This is one of the reasons that in some cases we use the alternative following concept to model behavior.

A *choice function* $C$ is required to assign to every nonempty $A \subseteq X$ a nonempty *subset* of $A$, that is, $C(A) \subseteq A$. According to our interpretation of a choice problem, a decision maker has to select a unique element from every choice set. Thus, $C(A)$ cannot be interpreted as the choice made by the decision maker when he has to make a choice from $A$. The revised interpretation of $C(A)$ is the *set* of all elements in $A$ that are satisfactory in the sense that the decision maker has no desire to move away from any of them. In other words, a choice function reflects an "internal equilibrium": If the decision maker facing $A$ considers an alternative outside $C(A)$, he will not continue searching for another alternative. If he happens to consider an alternative inside $C(A)$, he will take it.

We now define the induced choice function (assuming it is never empty) as $C_{\succsim}(A) = \{x \in A \mid x \succsim y \textit{ for all } y \in A\}$. Condition $*$ is now replaced by the condition that if $x$ is revealed to be at least as good as $y$ in one choice problem, $y$ will never be "chosen" without $x$ when $x$ is feasible:

### *The Weak Axiom of Revealed Preference (WA):*

We say that $C$ satisfies WA if whenever $x, y \in A \cap B$, $x \in C(A)$ and $y \in C(B)$, it is also true that $x \in C(B)$ (fig. 3.2). In other words, if $y$ is "chosen" while $x$ is available, then it will never be the case that $x$ is "chosen" without $y$ when both are available.

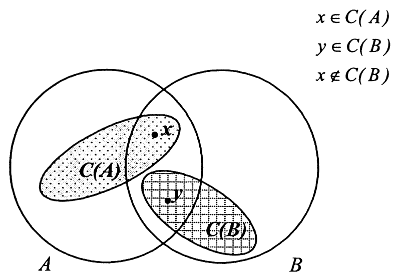



**Figure 3.2**
Violation of the weak axiom.

***Proposition:***

Assume that $C$ is a choice function with a domain that includes at least all subsets of size not greater than 3. Assume that $C$ satisfies WA. Then, there is a preference $\succsim$ so that $C = C_{\succsim}$.

***Proof:***

Define $x \succsim y$ if $x \in C(\{x, y\})$. We will now show that the relation is a preference:

- *Completeness*: Follows from $C(\{x, y\}) \neq \emptyset$.
- *Transitivity*: If $x \succsim y$ and $y \succsim z$ then $x \in C(\{x, y\})$ and $y \in C(\{y, z\})$. If $x \notin C(\{x, z\})$, then $C(\{x, z\}) = \{z\}$. By WA, $x \notin C(\{x, y, z\})$ (by WA $x$ cannot be revealed to be as good as $z$ because $z$ was chosen without $x$ from $\{x, z\}$). Similarly, $y \notin C(\{x, y, z\})$ (by WA, $y$ cannot be chosen without $x$ while $x \in C(\{x, y\})$). And also, $z \notin C(\{x, y, z\})$ (by WA, $z$ cannot be chosen without $y$ while $y \in C(\{y, z\})$). This contradicts the nonemptiness of $C(\{x, y, z\})$.

It remains to be shown that $C(B) = C_{\succsim}(B)$.

Assume that $x \in C(B)$ and $x \notin C_{\succsim}(B)$. That is, there is $y \in B$ so that $y$ is strictly better than $x$, or in other words, $C(\{x, y\}) = \{y\}$, thus contradicting WA.

Assume that $x \in C_{\succsim}(B)$ and $x \notin C(B)$. Let $y \in C(B)$. By WA $x \notin C(\{x, y\})$ and thus $C(\{x, y\}) = \{y\}$, and therefore $y \succ x$, contradicting $x \in C_{\succsim}(B)$.

### The Satisficing Procedure

The fact that we can present any choice function satisfying condition $*$ (or WA) as an outcome of the optimization of some preference relation is a key argument for the view that the scope of microeconomic models is much wider than that of the models in which agents carry out explicit optimization. But have we indeed expanded the scope of economic models beyond the circumstances in which decision makers carry out explicit optimization?

Consider the following "decision scheme," named *satisficing* by Herbert Simon. Let $v : X \to \Re$ be a valuation of the elements in $X$, and let $v^* \in \Re$ be a threshold of satisfaction. Let $O$ be an ordering of the alternatives in $X$. Given a set $A$, the decision maker arranges the elements of this set in a list $L(A, O)$ according to the ordering $O$. He then chooses the first element in $L(A, O)$ that has a $v$-value at least as large as $v^*$. If there is no such element in $A$, the decision maker chooses the last element in $L(A, O)$.

Let us show that the choice function induced by this procedure satisfies condition $*$. Assume that $a$ is chosen from $B$ and is also a member of $A \subset B$. The list $L(A, O)$ is obtained from $L(B, O)$ by eliminating all elements in $B - A$. If $v(a) \geq v^*$ then $a$ is the first satisfactory element in $L(B, O)$, and is also the first satisfactory element in $L(A, O)$. Thus $a$ is chosen from $A$. If all elements in $B$ are unsatisfactory, then $a$ must be the last element in $L(B, O)$. Since $A$ is a subset of $B$, all elements in $A$ are unsatisfactory and $a$ is the last element in $L(A, O)$. Thus, $a$ is chosen from $A$.

Note, however, that even a "small" variation in this scheme leads to a variation of the procedure such that it no longer satisfies $*$. For example:

*Satisficing using two orderings*: Let $X$ be a population of university graduates who are potential candidates for a job. Given a set of actual candidates, count their number. If the number is smaller than 5, order them alphabetically. If the number of candidates is above 5, order them by their social security number. Whatever ordering is used, choose the first candidate whose undergraduate average is above 85. If there are none, choose the last student on the list.

Condition $*$ is not satisfied. It may be that $a$ is the first candidate with a satisfactory grade in a long list of students ordered by their social security numbers. Still, $a$ might not be the first candidate with a satisfactory grade on a list of only three of the candidates appearing on the original list when they are ordered alphabetically.

The satisficing procedure, though it is stated in a way that seems unrelated to the maximization of a preference relation or utility function, can be described as if the decision maker maximizes a preference relation. I know of no other examples of interesting general schemes for choice procedures that satisfy $*$ other than the "rational man" and the satisficing procedures. However, later on, when we discuss consumer theory, we will come across several other appealing examples of demand functions that can be rationalized though they appear to be unrelated to the maximization of a preference relation.

## Psychological Motives Not Included within the Framework

The more modern attack on the standard approach to modeling economic agents comes from psychologists, notably from Amos Tversky and Daniel Kahneman. They have provided us with beautiful examples demonstrating not only that rationality is often violated, but that there are systematic reasons for the violation resulting from certain elements within our decision procedures. Here are a few examples of this kind that I find particularly relevant.

### *Framing*

The following experiment (conducted by Tversky and Kahneman 1986) demonstrates that the way in which alternatives are framed may affect decision makers' choices. Subjects were asked to imagine being confronted by the following choice problem:

An outbreak of disease is expected to cause 600 deaths in the US. Two mutually exclusive programs are expected to yield the following results:

a. 400 people will die.
b. With probability 1/3, 0 people will die and with probability 2/3, 600 people will die.

In the original experiment, a different group of subjects was given the same background information and asked to choose from the following alternatives:

c. 200 people will be saved.
d. With probability 1/3, all 600 will be saved and with probability 2/3, none will be saved.

While only 22% of the first group chose *a*, 72% of the second group chose *c*. My experience offering both questions to 170 graduate students in New York, Princeton, and Tel Aviv is similar even though they were the same students who responded to the two questions: 31% of the students chose *a* and 53% chose *c*.

These are "problematic" results since, by any reasonable criterion *a* and *c* are identical alternatives, as are *b* and *d*. Thus, the choice from $\{a, b\}$ should be consistent with the choice from $\{c, d\}$. The results expose the sensitivity of choice to the framing of the alternatives. What is more basic to rational decision making than taking the same choice when only the manner in which the problems are stated is different?

### *Simplifying the Choice Problem and the Use of Similarities*

The following experiment was also conducted by Tversky and Kahneman. One group of subjects was presented with the following choice:

Choose one of the two roulette games *a* or *b*. Your prize is the one corresponding to the outcome of the chosen roulette game as specified in the following tables:

| | Color | White | Red | Green | Yellow |
|---|---|---|---|---|---|
| (*a*) | Chance % | 90 | 6 | 1 | 3 |
| | Prize $ | 0 | 45 | 30 | −15 |

| | Color | White | Red | Green | Yellow |
|---|---|---|---|---|---|
| (*b*) | Chance % | 90 | 7 | 1 | 2 |
| | Prize $ | 0 | 45 | −10 | −15 |

A different group of subjects was presented the same background information and asked to choose between:

| | Color | White | Red | Green | Blue | Yellow |
|---|---|---|---|---|---|---|
| (*c*) | Chance % | 90 | 6 | 1 | 1 | 2 |
| | Prize $ | 0 | 45 | 30 | −15 | −15 |

and

(*d*)

| Color | White | Red | Green | Blue | Yellow |
|---|---|---|---|---|---|
| Chance % | 90 | 6 | 1 | 1 | 2 |
| Prize $ | 0 | 45 | 45 | −10 | −15 |

In the original experiment, 58% of the subjects in the first group chose *a*, while nobody in the second group chose *c*. I presented the two problems, one after the other, to 170 graduate students in New York, Princeton, and Tel Aviv: 43% chose *a* and 10% chose *c*. Interestingly, the median response time among the students who answered *a* was 60 seconds, whereas the median response time of the students who answered *b* was 91 seconds.

The results demonstrate a common procedure people practice when confronted with a complicated choice problem. We often transfer the complicated problem into a simpler one by "canceling" similar elements. While *d* clearly dominates *c*, the comparison between *a* and *b* is not as easy. Many subjects "cancel" the probabilities of Yellow and Red and are left with comparing the prizes of Green, a process that leads them to choose *a*.

Incidentally, several times in the past, when I presented these choice problems in class, I have had students (some of the best students, in fact) who chose *c*. They explained that they identified the second problem with the first and used the procedural rule: "I chose *a* from $\{a, b\}$. The alternatives *c* and *d* are identical to the alternatives *a* and *b*, respectively. It is only natural then, that I choose *c* from $\{c, d\}$." This observation brings to our attention a hidden facet of the rational-man model. The model does not allow a decision maker to employ a rule such as: "In the past I chose *x* from *B*. The choice problems *A* and *B* are similar. Therefore, I shall choose *x* from *A*."

### *Reason-Based Choice*

Making choices sometimes involves finding reasons to pick one alternative over the others. When the deliberation involves the use of reasons strongly associated with the problem at hand ("internal reasons"), we often find it difficult to reconcile the choice with the rational man paradigm.

Imagine, for example, a European student who would choose *Princeton* if allowed to choose from {*Princeton*, *LSE*} and would choose *LSE* if he had to choose from {*Princeton*, *Chicago*, *LSE*}. His explanation is that he prefers an American university so long as he does not have to choose between American schools—a choice he deems harder. Having to choose from {*Princeton*, *Chicago*, *LSE*}, he finds it difficult deciding between *Princeton* and *Chicago* and therefore chooses not to cross the Atlantic. His choice does not satisfy $*$, not because of a careless specification of the alternatives (as in the restaurant's menu example discussed previously), but because his reasoning involves an attempt to avoid the difficulty of making a decision.

Another example follows Huber, Payne, and Puto (1982):

Let $a = (a_1, a_2)$ be "a holiday package of $a_1$ days in Paris and $a_2$ days in London." Choose one of the four vectors $a = (7, 4)$, $b = (4, 7)$, $c = (6, 3)$, and $d = (3, 6)$.

All subjects in the experiment agreed that a day in Paris and a day in London are desirable goods. Some of the subjects were requested to choose between the three alternatives $a$, $b$, and $c$; others had to choose between $a$, $b$, and $d$. The subjects exhibited a clear tendency toward choosing $a$ out of the set $\{a, b, c\}$ and choosing $b$ out of the set $\{a, b, d\}$.

A related experiment is reported by Tversky and Shafir (1992): Subjects reviewed a list of twelve lotteries, including:

($x$) 65% chance to win \$15.
($y$) 30% chance to win \$35.
($z$) 65% chance to win \$14.

Afterwards, they were presented with a pair of lotteries; some got $x$ and $z$ and others $y$ and $z$. They had to either choose one of them or pay \$1 and receive an additional option. Significantly more subjects chose to pay the extra dollar when they had to choose between $x$ and $y$ than when they had to choose between $x$ and $z$.

To conclude, decision makers look for reasons to prefer one alternative over the other. Typically, making decisions by using "external reasons" (which do not refer to the properties of the choice set) will not cause violations of rationality. However, applying "internal reasons" such as "I prefer the alternative $a$ over the alternative $b$ since $a$ clearly dominates the other alternative $c$ while $b$ does not" might cause conflicts with condition $*$.

## Bibliographic Notes

*Recommended readings*: Kreps 1990, 24–30; Mas-Colell et al. 1995, chapter 1 C,D.

An excellent book on the lecture's subject is Kreps (1988). For the sources of consistency in choice and revealed preference assumptions, see Samuelson (1948), Houthakker (1950), and Richter (1966). Simon (1955) is the source of the discussion of satisficing. For a discussion of the bounded rationality approach to choice, see Rubinstein (1998). An excellent introduction to the Dutch Books arguments is Yaari (1985). Kahneman and Tversky (2000) is a definitive textbook on the psychological criticism of the economic approach to rationality. Rabin (1998) surveys the modern economics and psychology approach.

# Problem Set 3

**Problem 1.** (*Easy*)
Discuss the compatibility of the following "procedural elements" with the "rational man" paradigm:

a. The decision maker has in mind a ranking of all alternatives and chooses the alternative that is the worst according to this ranking.
b. The decision maker chooses an alternative with the intention that another person will suffer the most.
c. The decision maker asks his two children to rank the alternatives and then chooses the alternative that is the best "on average."
d. The decision maker has an ideal point in mind and chooses the alternative that is closest to the ideal point.
e. The decision maker looks for the alternative that appears most often in the choice set.
f. The decision maker always selects the first alternative that comes to his attention.
g. The decision maker searches for someone he knows who will choose an action that is feasible for him.
h. The decision maker orders all alternatives from left to right and selects the median.

**Problem 2.** (*Moderately difficult*)
Let us say that you have to make a choice from a set $A$. Does it matter whether (a) you make a choice from the entire set or (b) you first partition $A$ into the subsets $A_1$ and $A_2$, then make a selection from each of the sets and finally make a choice from the elements you selected from among $A_1$ and $A_2$?

a. Formulate a "path independence" property.
b. Show that the rational decision maker satisfies the property.
c. Find examples of choice procedures that do not satisfy this property.
d. Show that if a (single-valued) choice function satisfies path independence, then it is consistent with rationality.
e. Assume that $C$ is a (multivalued) choice function satisfying path independence. Can it be rationalized by a preference relation?

**Problem 3.** (*Easy*)

Check whether the following two choice functions satisfy WA:

$C(A) = \{x \in A|$ *the number of* $y \in X$ *for which* $V(x) \geq V(y)$ *is at least* $|X|/2\}$, and if the set is empty then $C(A) = A$.

$D(A) = \{x \in A|$ *the number of* $y \in A$ *for which* $V(x) \geq V(y)$ *is at least* $|A|/2\}$ .

**Problem 4.** (*Moderately difficult*)

Consider the following choice procedure. A decision maker has a strict ordering $\succsim$ over the set $X$ and he assigns to each $x \in X$ a natural number *class*$(x)$ interpreted as the "class" of $x$. Given a choice problem $A$ he chooses the element in $A$ that is the best among those elements in $A$, that belong to the "most popular" class in $A$ (that is, the class that appears in $A$ most often). If there is more than one most popular class, he picks the best element from the members of $A$ that belong to a most popular class with the highest class number.

a. Is the procedure consistent with the "rational man" paradigm?
b. Can every choice function be "explained" as an outcome of such a procedure?

(Try to formalize a "property" that is satisfied by such choice procedures and is clearly not satisfied by some other choice functions.)

**Problem 5.** (*Moderately difficult.* Based on Kalai, Rubinstein, and Spiegler 2002)

Consider the following two choice procedures. Explain the procedures and try to persuade a skeptic that they "make sense." Determine for each of them whether they are consistent with the rational-man model.

a. The primitives of the procedure are two numerical (one-to-one) functions $u$ and $v$ defined on $X$ and a number $v^*$. For any given choice problem $A$, let $a^* \in A$ be the maximizer of $u$ over $A$, and let $b^*$ be the maximizer of $v$ over the set $A$. The decision maker chooses $a^*$ if $v(a^*) \geq v^*$ and chooses $b^*$ if $v(a^*) < v^*$.
b. The primitives of the procedure are two numerical (one-to-one) functions $u$ and $v$ defined on $X$ and a number $u^*$. For any given choice problem $A$, the decision maker chooses the element $a^* \in A$ that maximizes $u$ if $u(a^*) \geq u^*$, and $v$ if $u(a^*) < u^*$.

**Problem 6.** (*Moderately difficult*)

The standard economic choice model assumes that choice is made from a *set*. Let us construct a model where the choice is assumed to be from a *list*.

Let $X$ be a finite "grand set." A *list* is a nonempty finite vector of elements in $X$. In this problem, consider a *choice function* $C$ to be a function that assigns to each vector $L =< a_1, \ldots, a_K >$ a single element from $\{a_1, \ldots, a_K\}$. (Thus, for example, the list $< a, b >$ is distinct from $< a, a, b >$ and $< b, a >$). For all $L_1, \ldots, L_m$ define $< L_1, \ldots, L_m >$ to be the list that is the concatenation of the $m$ lists. (Note that if the length of $L_i$ is $k_i$ , the length of the concatenation is $\Sigma_{i=1,\ldots,m} k_i$). We say that $L'$ *extends* the list $L$ if there is a list $M$ such that $L' =< L, M >$.

We say that a choice function $C$ satisfies property $I$ if for all $L_1, \ldots, L_m$ $C(< L_1, \ldots, L_m >) = C(< C(L_1), \ldots, C(L_m) >)$.

a. Interpret property $I$. Give two (distinct) examples of choice functions that satisfy $I$ and two examples of choice functions which do not.
b. Define formally the following two properties of a choice function:

   - *Order Invariance*: A change in the order of the elements of the list does not alter the choice.
   - *Duplication Invariance*: Deleting an element that appears in the list elsewhere does not change the choice.

c. Characterize the choice functions that satisfy Order Invariance, Duplication Invariance, and condition $I$.
d. Assume now that in the back of the decision maker's mind is a value function $u$ defined on the set $X$ (such that $u(x) \neq u(y)$ for all $x \neq y$). For any choice function $C$ define $v_C(L) = u(C(L))$.

We say that $C$ *accommodates a longer list* if whenever $L'$ extends $L$, $v_C(L') \geq v_C(L)$ and there is a list $L'$ which extends a list $L$ for which $v_C(L') > v_C(L)$.

e. Give two interesting examples of choice functions that accommodate a longer list.
f. Give two interesting examples of choice functions which satisfy property $I$ but which do not accommodate a longer list.

**Problem 7.** (*Reading*)
Read Sen (1993). Invent two sound choice procedures and discuss their relation to the "rational man" paradigm.

LECTURE 4

# Consumer Preferences

## The Consumer's World

Up to this point we have dealt with the basic economic model of rational choice. In this lecture we will discuss a special case of the rational man paradigm: the *consumer*. A consumer is an economic agent who makes choices between available combinations of commodities. As usual, we have a certain image in mind: a person goes to the marketplace with money in hand and comes back with a bundle of commodities.

As before, we will begin with a discussion of consumer preferences and utility, and only then discuss consumer choice. Our first step is to move from an abstract treatment of the set $X$ to a more detailed structure. We take $X$ to be $\Re^K_+ = \{x = (x_1, \ldots, x_K) |$ for all $k$, $x_k \geq 0\}$. An element of $X$ is called a *bundle*. *A bundle x* is interpreted as a combination of $K$ commodities where $x_k$ is the quantity of commodity $k$.

Given this special interpretation of $X$, we impose some conditions on the preferences in addition to those assumed for preferences in general. The additional three conditions use the structure of the space $X$: monotonicity uses the orderings on the axis (the ability to compare bundles by the amount of any particular commodity); continuity uses the topological structure (the ability to talk about closeness); convexity uses the algebraic structure (the ability to speak of the sum of two bundles and the multiplication of a bundle by a scalar).

## Monotonicity

Monotonicity is a property that gives commodities the meaning of "goods." It is the condition that more is better. Increasing the amount of some commodities cannot hurt, and increasing the amount of all commodities is strictly desired. Formally,

***Monotonicity:***

The relation $\succsim$ satisfies *monotonicity* if for all $x, y \in X$,
if $x_k \geq y_k$ for all $k$, then $x \succsim y$, and
if $x_k > y_k$ for all $k$, then $x \succ y$.

In some cases, we will further assume that the consumer is strictly happier with any additional quantity of any commodity.

***Strong Monotonicity:***

The relation $\succsim$ satisfies *strong monotonicity* if for all $x, y \in X$
if $x_k \geq y_k$ for all $k$ and $x \neq y$, then $x \succ y$.

Of course, in the case that preferences are represented by a utility function, preferences satisfying monotonicity (or strong monotonicity) are represented by monotonic increasing (or strong monotonic increasing) utility functions.

***Examples:***

- The preference represented by $min\{x_1, x_2\}$ satisfies *monotonicity* but not *strong monotonicity*.
- The preference represented by $x_1 + x_2$ satisfies *strong monotonicity*.
- The preference relation $|x - x^*|$ satisfies *nonsatiation*, a related property that is sometimes used in the literature: for every $x \in X$ and for any $\varepsilon > 0$ there is some $y \in X$ that is less than $\varepsilon$ away from $x$ so that $y \succ x$. Every monotonic preference relation satisfies nonsatiation, but the reverse is, of course, not true.

## Continuity

We will use the topological structure of $\Re_+^K$ (induced from the standard distance function $d(x, y) = \sqrt{\sum (x_k - y_k)^2}$ ) to apply the definition of continuity discussed in Lecture 2. We say that the preferences

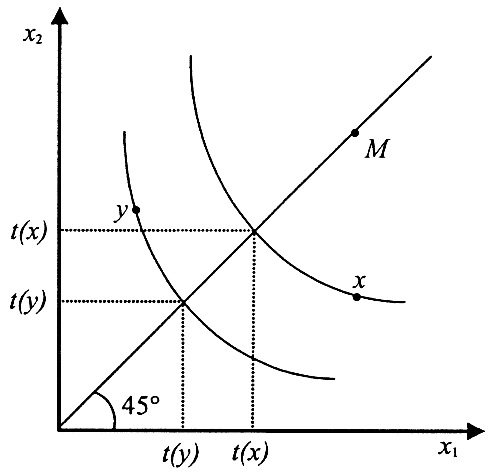



**Figure 4.1**

$\succsim$ satisfy *continuity* if for all $a, b \in X$, if $a \succ b$, then there is an $\varepsilon > 0$ such that $x \succ y$ for any $x$ and $y$ such that $d(x, a) < \varepsilon$ and $d(y, b) < \varepsilon$.

### Existence of a Utility Representation

Debreu's theorem guarantees that any continuous preference relation is represented by some (continuous) utility function. If we assume monotonicity as well, we then have a simple and elegant proof:

***Claim:***

Any consumer preference relation satisfying *monotonicity* and continuity can be represented by a utility function.

***Proof:***

Let us first show that for every bundle $x$, there is a bundle on the main diagonal (having equal quantities of all commodities), such that the consumer is indifferent between that bundle and the bundle $x$. (See fig. 4.1.) The bundle $x$ is at least as good as the bundle $0 = (0, \ldots, 0)$. On the other hand, the bundle

$M = (max_k\{x_k\}, \ldots, max_k\{x_k\})$ is at least as good as $x$. Both 0 and $M$ are on the main diagonal. By continuity, there is a bundle on the main diagonal that is indifferent to $x$ (see the problem set). By monotonicity this bundle is unique; we will denote it by $(t(x), \ldots, t(x))$. Let $u(x) = t(x)$. To see that the function $u$ represents the preferences, note that by transitivity of the preferences $x \succsim y$ iff $(t(x), \ldots, t(x)) \succsim (t(y), \ldots, t(y))$, and by *monotonicity* this is true iff $t(x) \geq t(y)$.

## Convexity

Consider, for example, a scenario in which the alternatives are candidates for some position and are ranked in a left-right array as follows:

—–*a*—*b*—–*c*—–*d*——*e*—.

In normal circumstances, if we know that a voter prefers $b$ to $d$, then:

- We tend to conclude that $c$ is preferred to $d$, but not necessarily that $a$ is preferred to $d$ (the candidate $a$ may be too extreme).
- We tend to conclude that $d$ is preferred to $e$ (namely, we do not find it plausible that both $e$ and $b$ are preferable to $d$).

Convexity is meant to capture related intuitions that rely on the existence of "geography" in the sense that we can talk about an alternative being between two other alternatives. The convexity assumption is appropriate for a situation in which the argument "if a move from $d$ to $b$ is an improvement then so is a move part of the way to $c$" is legitimate, while the argument "if a move from $b$ to $d$ is harmful then so is a move part of the way to $c$" is not.

Following are two formalizations of these two intuitions (fig. 4.2). We will see that they are equivalent.

### *Convexity 1:*

The preference relation $\succsim$ satisfies *convexity 1* if $x \succsim y$ and $\alpha \in (0, 1)$ implies that $\alpha x + (1 - \alpha)y \succsim y$.

This captures the intuition that if $x$ is preferred to $y$, then "going a part of the way from $y$ to $x$" is also an improvement.

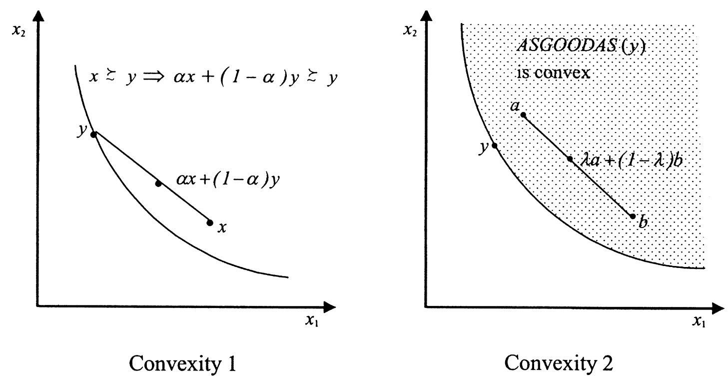



**Figure 4.2**
Two definitions of convexity.

***Convexity 2:***

The relation $\succsim$ satisfies *convexity 2* if for all $y$, the set $AsGood(y) = \{z \in X | z \succsim y\}$ is convex.

(Recall that a set $A$ is convex if for all $a, b \in A$ and for all $\lambda \in [0, 1]$, $\lambda a + (1 - \lambda)b \in A$.) This captures the intuition that if both $z_1$ and $z_2$ are better than $y$, then the average of $z_1$ and $z_2$ is definitely better than $y$.

***Claim:***

A preference $\succsim$ satisfies convexity 1 if and only if it satisfies convexity 2.

***Proof:***

Assume that $\succsim$ satisfies convexity 1 and let $a \succsim y$ and $b \succsim y$ ; without loss of generality assume $a \succsim b$. Then by the definition of convexity 1, $\lambda a + (1 - \lambda)b \succsim b$ and by the transitivity of $\succsim$, $\lambda a + (1 - \lambda)b \succsim y$ and thus $\lambda a + (1 - \lambda)b \in AsGood(y)$.

Assume that $\succsim$ satisfies convexity 2. If $x \succsim y$ then both $x$ and $y$ are in $AsGood(y)$ and thus $\alpha x + (1 - \alpha)y \in AsGood(y)$, which means that $\alpha x + (1 - \alpha)y \succsim y$.

As usual, the above property also has a stronger version:

***Strict Convexity:***

The preference relation $\succsim$ satisfies *strict convexity* if for every $a \succsim y$, $b \succsim y$, $a \neq b$ and $\lambda \in (0, 1)$ imply that $\lambda a + (1 - \lambda)b \succ y$.

***Example:***

The preferences represented by $\sqrt{x_1} + \sqrt{x_2}$ satisfy strict convexity. The preferences represented by $min\{x_1, x_2\}$ and $x_1 + x_2$ satisfy convexity but not strict convexity. The lexicographic preferences satisfy strict convexity. The preferences represented by $x_1^2 + x_2^2$ do not satisfy convexity.

We now look at the properties of the utility representations of convex preferences.

***Quasi-Concavity:***

A function $u$ is *quasi-concave* if for all $y$ the set $\{x| u(x) \geq u(y)\}$ is convex.

The term's name derives from the fact that for any concave function $f$ and for any $y$ the set $\{x|f(x) \geq f(y)\}$ is convex.

Obviously, if a preference relation is represented by a utility function, then it is convex iff the utility function is quasi-concave. However, the convexity of $\succsim$ does not imply that a utility function representing $\succsim$ is concave. (Recall that $u$ is *concave* if for all $x$, $y$, and $\lambda \in [0, 1]$, we have $u(\lambda x + (1 - \lambda)y) \geq \lambda u(x) + (1 - \lambda)u(y)$.)

**Special Classes of Preferences**

Often in economics, we limit our discussion of consumer preferences to a class of preferences possessing some additional special properties. Following are some examples of "popular" classes of preference relations discussed in the literature.

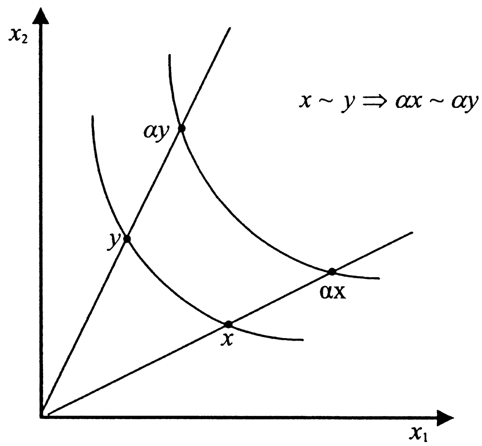



**Figure 4.3**
Homothetic preferences.

***Homothetic Preferences:***

A preference $\succsim$ is *homothetic* if $x \succsim y$ implies $\alpha x \succsim \alpha y$ for all $\alpha \geq 0$. (See fig. 4.3.)

The preferences represented by $\Pi_{k=1,\ldots,K} x_k^{\beta_k}$, where $\beta_k$ is positive, are homothetic. In fact, any preference relation represented by a utility function $u$ that is homogeneous of any degree $\lambda$ is homothetic. ($x \succsim y$ iff $u(x) \geq u(y)$ iff $\alpha^\lambda u(x) \geq \alpha^\lambda u(y)$ iff $u(\alpha x) \geq u(\alpha y)$ iff $\alpha x \succsim \alpha y$).

Note that lexicographic preferences are also homothetic.

***Claim:***

Any homothetic, continuous, and monotonic preference relation on the commodity bundle space can be represented by a utility function that is homogeneous of degree one.

***Proof:***

We have already proven that any bundle $x$ has a unique bundle $(t(x), \ldots, t(x))$ on the main diagonal so that $x \sim (t(x), \ldots, t(x))$, and that the function $u(x) = t(x)$ represents $\succsim$. By the assumption that

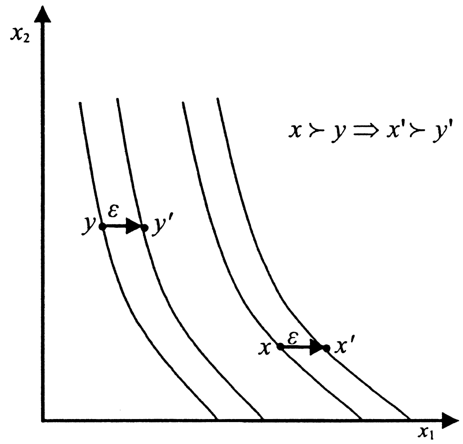



**Figure 4.4**
Quasi-linear (in good 1) preferences.

the preferences are homothetic, $\alpha x \sim (\alpha t(x), \ldots, \alpha t(x))$ and thus $u(\alpha x) = \alpha t(x) = \alpha u(x)$.

Let us now consider an additional class of consumer preferences.

***Quasi-Linear Preferences:***

A preference is *quasi-linear* in commodity 1 (referred to as the "numeraire") if $x \succsim y$ implies $(x + \varepsilon e_1) \succsim (y + \varepsilon e_1)$ (where $e_1 = (1, 0, \ldots, 0)$ and $\varepsilon > 0$). (See fig. 4.4.)

The indifference curves of preferences that are quasi-linear in commodity 1 are parallel to each other (relative to the first commodity axis). That is, if $I$ is an indifference curve, then the set $I_\varepsilon = \{x |$ there exists $y \in I$ such that $x = y + (\varepsilon, 0, \ldots, 0)\}$ is an indifference curve. Any preference relation represented by $x_1 + v(x_2, \ldots, x_K)$ for some function $v$ is quasi-linear in commodity 1. Furthermore:

***Claim:***

Any continuous preference relation satisfying strong monotonicity (at least in commodity 1) and quasi-linearity in commodity 1 can be represented by a utility function of the form $x_1 + v(x_2, \ldots, x_K)$.

***Proof:***

In the problem set you will prove that every preference relation that is monotonic, continuous, and quasi-linear in commodity 1 satisfies that for every $(x_2, \ldots, x_K)$ there is some number $v(x_2, \ldots, x_K)$ so that $(v(x_2, \ldots, x_K), 0, \ldots, 0) \sim (0, x_2, \ldots, x_K)$. Then, from quasi-linearity in commodity 1, for every bundle $x$, $(x_1 + v(x_2, \ldots, x_K), 0, \ldots, 0) \sim (x_1, x_2, \ldots, x_K)$, and thus by strong monotonicity in the first commodity, the function $x_1 + v(x_2, \ldots, x_K)$ represents $\succsim$ .

## Differentiable Preferences (and the Use of Derivatives in Economic Theory)

We often assume in microeconomics that utility functions are differentiable and thus use standard calculus to analyze the consumer. In this course I (almost) avoid calculus. This is part of a deliberate attempt to steer you away from a "mechanistic" approach to economic theory.

Can we give the differentiability of a utility function an "economic" interpretation? We introduce a nonconventional definition of *differentiable preferences*. Basically, differentiability of preferences requires that the directions for improvement can be described using "local prices."

Let us confine ourselves to preferences satisfying monotonicity and convexity. For any vector $x$ we say that the direction of change $d \in \Re^K$ is an *improvement* direction at $x$ if there is some $\varepsilon > 0$ so that $x + \varepsilon d \succ x$. In other words, there is some move from $x$ in the direction of $d$, which is an improvement. Let $D(x)$ be the set of all improvement directions at $x$. Given monotonicity, $D(x)$ includes all positive vectors. We say that a consumer's monotonic preferences $\succsim$ are *differentiable at the bundle* $x$ if there is a vector $v(x)$ of $K$ nonnegative numbers so that $D(x)$ contains all vectors $d \in \Re^K$ for which $dv(x) > 0$ ($dv(x)$ is the inner product of $d$ and $v(x)$). In such a case the vector of numbers $(v_1(x), \ldots, v_K(x))$ is interpreted as the vector of "subjective values" of the commodities. Starting from $x$, any small-enough move in a direction that is evaluated by this vector as positive is an improvement. We say that $\succsim$ is *differentiable* if it is differentiable at any bundle $x$ (see fig. 4.5).

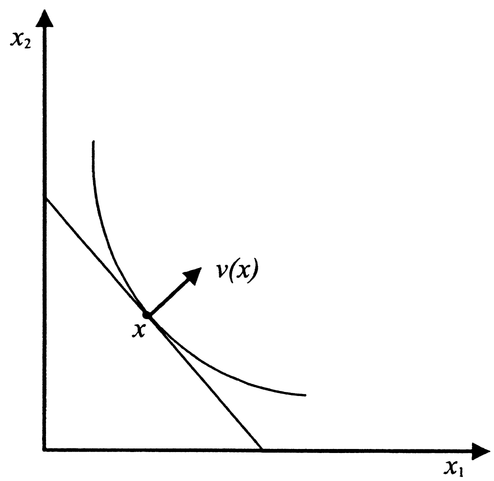



**Figure 4.5**
Differentiable preferences.

**Examples:**

- The preferences represented by $2x_1 + 3x_2$ are differentiable. At each point $x$, $v(x) = (2, 3)$.
- The preferences represented by $\min\{x_1, \ldots, x_K\}$ are differentiable only at points where there is a unique commodity $k$ for which $x_k < x_l$ for all $l \neq k$ (verify). For example, at $x = (5, 3, 8, 6), v(x) = (0, 1, 0, 0)$.

Assume $u$ is a differentiable quasi-concave utility function representing the consumer's preferences. Let $du/dx_k(x)$ be the partial derivative of $u$ with respect to the commodity $k$ at point $x$. If all vectors $(du/dx_k(x))$ of partial derivatives are nonzero, then the induced preference is differentiable with $v_k(x) = du/dx_k(x)$ (the partial derivative of $u$ with respect to the commodity $k$ at the point $x$).

## Bibliographic Notes

*Recommended readings*: Kreps 1990, 32–37; Mas-Colell et al. 1995, Chapter 3, A–C.

The material in this lecture up to the discussion of differentiability is fairly standard and closely parallels that found in Arrow and Hahn (1971).

# Problem Set 4

**Problem 1.** (*Easy*)
Characterize the preference relations on the interval $[0, 1]$ that are continuous and strictly convex.

**Problem 2.** (*Easy*)
Show that if the preferences $\succsim$ satisfy continuity and $x \succsim y \succsim z$, then there is a bundle $m$ on the interval connecting $x$ and $z$ such that $y \sim m$.

**Problem 3.** (*Moderate*)
Show that if the preferences $\succsim$ satisfy continuity and monotonicity, then the function $u(x)$, defined by $x \sim (u(x), \dots, u(x))$, is continuous.

**Problem 4.** (*Moderate*)
In a world with two commodities, consider the following condition:

The preference relation $\succsim$ satisfies *convexity 3* if for all $x$ and $\varepsilon$

$$(x_1, x_2) \sim (x_1 - \varepsilon, x_2 + \delta_1) \sim (x_1 - 2\varepsilon, x_2 + \delta_1 + \delta_2) \text{ implies } \delta_2 \geq \delta_1.$$

Interpret convexity 3 and show that for strong monotonic and continuous preferences, it is equivalent to the convexity of the preference relation.

**Problem 5.** (*Moderate*)
Formulate and prove a proposition of the following type:
If the preferences $\succsim$ are quasi linear in all commodities, continuous, and strongly monotonic, then there is a utility function of the form (... add a condition here ...) that represents it.

**Problem 6.** (*Difficult*)
Show that for any consumer's preference relation $\succsim$ satisfying continuity, monotonicity and *quasi-linearity with respect to commodity 1* and for every vector $x$, there is a number $v(x)$ so that $x \sim (v(x), 0, \dots, 0)$.

**Problem 7.** (*Easy*)

We say that a preference relation satisfies *separability* if it can be represented by an additive utility function, that is, a function of the type $u(x) = \Sigma_k v_k(x_k)$. Show that such preferences satisfy that for any subset of commodities $J$, and for any bundles $a$, $b$, $c$, $d$, we have

$$\left(a_J, c_{-J}\right) \succsim \left(b_J, c_{-J}\right) \Leftrightarrow \left(a_J, d_{-J}\right) \succsim \left(b_J, d_{-J}\right),$$

where $\left(x_J, y_{-J}\right)$ is the vector that takes the components of $x$ for any $k \in J$ and takes the components of $y$ for any $k \notin J$.

Demonstrate this condition geometrically for $K = 2$.

**Problem 8.** (*Moderate*)

Let $\succsim$ be monotonic and convex preferences that are represented by a differentiable utility function $u$.

- Show that for every $x$ there is a vector $v(x)$ of $K$ nonnegative numbers so that $d$ is an improvement at $x$ iff $dv(x) > 0$ ($dv(x)$ is the inner product of $v(x)$).
- Show that the preferences represented by the function $\min\{x_1, \ldots, x_K\}$ cannot be represented by a differentiable utility function.
- Check the differentiability of the lexicographic preferences in $\Re^2$.
- Assume that for any $x$ and for any $d \in D(x)$, $(x + d) \succ x$. What can you say about $\succsim$ ?

LECTURE 5

# Demand: Consumer Choice

## The Rational Consumer's Choice from a Budget Set

In Lecture 4 we discussed the consumer's preferences. In this lecture we adopt the "rational man" paradigm in discussing consumer choice.

Given a consumer's preference relation $\succsim$ on $X = \Re^K_+$, we can talk about his choice from any set of bundles. However, since we are laying the foundation for "price models," we are interested in the consumer's choice in a particular class of choice problems called budget sets. A *budget set* is a set of bundles that can be represented as $B(p, w) = \{x \in X \mid px \le w\}$, where $p$ is a vector of positive numbers (interpreted as prices) and $w$ is a positive number (interpreted as the consumer's wealth).

Obviously, any set $B(p, w)$ is compact (it is closed since it is defined by weak inequalities, and bounded since for any $x \in B(p, w)$ and for all $k$, $0 \le x_k \le w/p_k$). It is also convex since if $x, y \in B(p, w)$, then $px \le w$ , $py \le w$, $x_k \ge 0$, and $y_k \ge 0$ for all $k$. Thus, for all $\alpha \in [0, 1]$, $p[\alpha x + (1-\alpha)y] = \alpha px + (1-\alpha)py \le w$ and $\alpha x_k + (1-\alpha)y_k \ge 0$ for all $k$, that is, $\alpha x + (1-\alpha)y \in B(p, w)$.

We will refer to the problem of finding the $\succsim$-best bundle in $B(p, w)$ as the *consumer's problem*.

### *Claim:*

If $\succsim$ is a continuous relation, then all consumer problems have a solution.

### *Proof:*

If $\succsim$ is continuous, then it can be represented by a continuous utility function $u$. By the definition of the term "utility representation," finding an $\succsim$ optimal bundle is equivalent to solving the problem $max_{x \in B(p,w)} u(x)$. Since the budget set is compact and $u$ is continuous, the problem has a solution.

To emphasize that a utility representation is not necessary for the current analysis, let us study a direct proof of the previous claim, avoiding the notion of utility.

***Direct Proof:***

For any $x \in B(p, w)$ define the set $Inferior(x) = \{y \in B(p, w) | x \succ y\}$. By the continuity of the preferences, every such set is open. Assume there is no solution to the consumer's problem of maximizing $\succsim$ on $B(p, w)$. Then, every $z \in B(p, w)$ is a member of some set $Inferior(x)$, that is, the collection of sets $\{Inferior(x) | \ x \in X\}$ covers $B(p, w)$. A collection of open sets that covers a compact set has a finite subset of sets that covers it. Thus, there is a finite collection $Inferior(x^1), \ldots,$ $Inferior(x^n)$ that covers $B(p, w)$. Letting $x^j$ be the optimal bundle within the finite set $\{x^1, \ldots, x^n\}$, we obtain that $x^j$ is an optimal bundle in $B(p, w)$, a contradiction.

***Claim:***

If $\succsim$ is convex, then the set of solutions for a choice from $B(p, w)$ (or any other convex set) is convex.

***Proof:***

If both $x$ and $y$ maximize $\succsim$ given $B(p, w)$, then $\alpha x + (1 - \alpha)y \in B(p, w)$ and, by the convexity of the preferences, $\alpha x + (1 - \alpha)y \succsim x \succsim z$ for all $z \in B(p, w)$. Thus, $\alpha x + (1 - \alpha)y$ is also a solution to the consumer's problem.

***Claim:***

If $\succsim$ is strictly convex, then every consumer's problem has at most one solution.

***Proof:***

Assume that both $x$ and $y$ (where $x \neq y$) are solutions to the consumer's problem $B(p, w)$. Then $x \sim y$ (both are solutions to the same maximization problem) and $\alpha x + (1 - \alpha)y \in B(p, w)$ (the budget set is convex). By the strict convexity of $\succsim$, $\alpha x + (1 - \alpha)y \succ x$, which is a contradiction of $x$ being a maximal bundle in $B(p, w)$.

### The Consumer's Problem with Differentiable Preferences

When the preferences are differentiable, we are provided with a "useful" condition for characterizing the optimal solution.

***Claim:***

If $x^*$ is an optimal bundle in the consumer problem and $k$ is a consumed commodity (i.e., $x_k^* > 0$), then it must be that $v_k(x^*)/p_k \geq v_j(x^*)/p_j$ for all other $j$, where $v_k(x^*)$ are the "subjective value numbers" (see the definition of differentiable preferences in Lecture 4).

***Proof:***

Assume that $x^*$ is a solution to the consumer's problem $B(p, w)$ and that $x_k^* > 0$ and $v_k(x^*)/p_k < v_j(x^*)/p_j$ (see fig. 5.1). A "move" in the direction of reducing the consumption of the $k$-th commodity by 1 and increasing the consumption of the $j$-th commodity by $p_k/p_j$ is an improvement since $v_j(x^*)p_k/p_j - v_k(x^*) > 0$. As $x_k^* > 0$, we can find $\varepsilon > 0$ small enough such that decreasing $k$'s quantity by $\varepsilon$ and increasing $j$'s quantity by $\varepsilon p_k/p_j$ is feasible. This brings the consumer to a strictly better bundle, contradicting the assumption that $x^*$ is a solution to the consumer's problem.

For the case in which the preferences are represented by a utility function $u$, we have $v_k(x^*) = \partial u/\partial x_k(x^*)$. In other words, the "value per dollar" at the point $x^*$ of the $k$-th commodity (which is consumed) must be as large as the "value per dollar" of any other commodity.

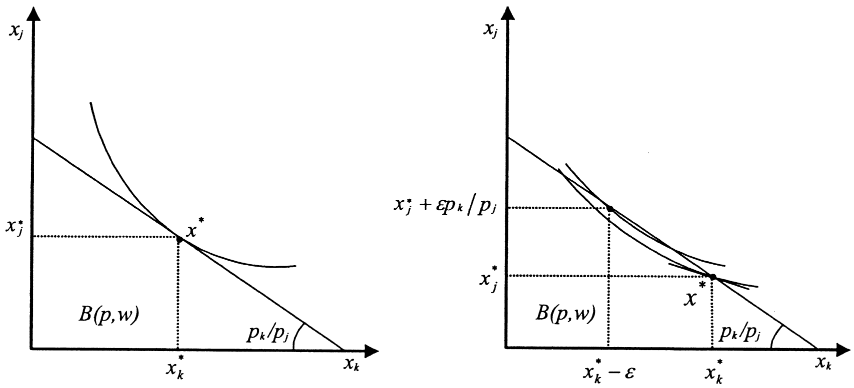



**Figure 5.1**
(a) $x^*$ is a solution to the consumer problem $B(p, w)$.
(b) $x^*$ is not a solution to the consumer problem $B(p, w)$.

From the above we can derive the "classic" necessary conditions on the consumer's maximization:

***Claim:***

If $x^*$ is a solution to the consumer's problem $B(p, w)$ and both $x_k^* > 0$ and $x_j^* > 0$, then the ratio $v_k(x^*)/v_j(x^*)$ must be equal to the price ratio $p_k/p_j$.

In order to establish sufficient conditions for maximization, we require also that the preferences be convex.

***Claim:***

If $\succsim$ is monotonic, convex, continuous, and differentiable, and if at $x^*$

- $px^* = w$,
- for all $k$ such that $x_k^* > 0$, and for any commodity $l$, $v_k(x^*)/p_k \geq v_l(x^*)/p_l$,

then $x^*$ is a solution to the consumer's problem.

***Proof:***

If $x^*$ is not a solution, then there is a bundle $z$ such that $pz \leq px^*$ and $z \succ x^*$. By continuity and monotonicity, there is a bundle $y \neq z$, with $y_k \leq z_k$ such that $y \succ x^*$ and $py < pz \leq px^*$. By convexity, any small move in the direction $(y - x^*)$ is an improvement and by differentiability, $v(x^*)(y - x^*) > 0$.

Let $\mu = v_k(x^*)/p_k$ for all $k$ with $x_k^* > 0$. Now,

$$0 > p(y - x^*) = \sum p_k(y_k - x_k^*) \geq \sum v_k(x^*)(y_k - x_k^*)/\mu$$

(since for a good with $x_k^* > 0$ we have $p_k = v_k(x^*)/\mu$, and for a good $k$ with $x_k^* = 0$, $(y_k - x_k^*) \geq 0$ and $p_k \geq v_k(x^*)/\mu$.) Thus, $0 \geq v(x^*)(y - x^*)$, a contradiction.

## The Demand Function

We have arrived at an important stage on the way to developing a market model in which we derive demand from preferences. Assume that the consumer's preferences are such that for any $B(p, w)$, the consumer's problem has a unique solution. Let us denote this solution by $x(p, w)$. The function $x(p, w)$ is called the *demand function*. The domain of the demand function is $\Re^{K+1}_{++}$ whereas its range is $\Re^K_+$.

***Example:***

Consider a consumer in a world with two commodities having the following lexicographic preference relation, attaching the first priority to the sum of the quantities of the goods and the second priority to the quantity of commodity 1:
$x \succsim y$ if $x_1 + x_2 > y_1 + y_2$ or both $x_1 + x_2 = y_1 + y_2$ and $x_1 \geq y_1$.

This preference relation is strictly convex but not continuous. It induces the following noncontinuous demand function:

$$x((p_1, p_2), w) = \begin{cases} (0, w/p_2) & \textit{if} \quad p_2 < p_1 \\ (w/p_1, 0) & \textit{if} \quad p_2 \geq p_1 \end{cases} .$$

We now turn to studying some properties of the demand function.

***Claim:***

$x(p, w) = x(\lambda p, \lambda w)$ (i.e., the demand function is *homogeneous of degree zero*).

***Proof:***

This follows (with no assumptions about the preference relations) from the basic equality $B(\lambda p, \lambda w) = B(p, w)$ and the assumption that the behavior of the consumer is "a choice from a set."

Note that this claim is sometimes interpreted as implying that "uniform inflation does not matter." This is an incorrect interpretation. We assumed, rather than concluded, that choice is made from a set independently of the way that the choice set is framed. Inflation can affect choice since behavior may be sensitive to the nominal prices and wealth even if the budget set is unchanged.

***Claim (Walras's law):***

If the preferences are monotonic, then any solution $x$ to the consumer's problem $B(p, w)$ is located on its budget curve (and thus, $px(p, w) = w$).

***Proof:***

If not, then $px < w$. There is an $\varepsilon > 0$ such that $p(x_1 + \varepsilon, \dots, x_K + \varepsilon) < w$. By monotonicity, $(x_1 + \varepsilon, \dots, x_K + \varepsilon) \succ x$, thus contradicting the assumption that $x$ is optimal in $B(p, w)$.

***Claim:***

If $\succsim$ is a continuous preference, then the demand function is continuous in prices (and also in $w$, see problem set).

***Proof:***

Once again, we could use the fact that the preferences have a continuous utility representation and apply a standard "maximum theorem." (If the function $f(x, a)$ is continuous, then the function $h(a) = argmax_x f(x, a)$ is continuous.) However, I prefer to present a proof that does not use the notion of a utility function:

Assume not. Then, there is a sequence of price vectors $p^n$ converging to $p^*$ such that $x(p^*, w) = x^*$, and $x(p^n, w)$ does not converge to $x^*$. Thus, we can assume that $(p^n)$ is a sequence converging to $p^*$ such that for all $n$ the distance $d(x(p^n, w), x^*) > \varepsilon$ for some positive $\varepsilon$.

All numbers $p^n_k$ are greater than some positive number $m$. Therefore, all vectors $x(p^n, w)$ belong to some compact set (the hypercube of bundles with no quantity above $w/m$) and thus, without loss of generality, we can assume that $x(p^n, w) \to y^*$ for some $y^* \neq x^*$.

Since $p^n x(p^n, w) \leq w$ for all $n$, it must be that $p^* y^* \leq w$, that is, $y^* \in B(p^*, w)$. Since $x^*$ is the unique solution for $B(p^*, w)$, we have $x^* \succ y^*$. By the continuity of the preferences, there are neighborhoods of $x^*$ and $y^*$ in which the strict preference is preserved. For sufficiently large $n$, $x(p^n, w)$ is in that neighborhood of $y^*$. Choose a bundle $z^*$ in the neighborhood of $x^*$ so that $p^* z^* < w$. For all sufficiently large $n$, $p^n z^* < w$; however, $z^* \succ x(p^n, w)$, which is a contradiction.

## Rationalizable Demand Functions

As in the general discussion of choice, we will now examine whether choice procedures are consistent with the rational man model. We can think of various possible definitions of rationalization.

One approach is to look for a preference relation (without imposing any restrictions that fit the context of the consumer) such that the chosen element from any budget set is the unique bundle maximizing the preference relation in that budget set. Thus, we say that the preferences $\succsim$ *fully rationalize* the demand function $x$ if for any $(p, w)$ the bundle $x(p, w)$ is the unique $\succsim$ maximal bundle within $B(p, w)$.

Alternatively, we could say that "being rationalizable" means that there are preferences such that the consumer's behavior is consistent with maximizing those preferences, that is, for any $(p, w)$ the bundle $x(p, w)$ is a $\succsim$ maximal bundle (not necessarily unique) within

$B(p, w)$. This definition is "empty" since any demand function is consistent with maximizing the "total indifference" preference. This is why we usually say that the preferences $\succsim$ *rationalize* the demand function $x$ if they are *monotonic* and for any $(p, w)$, the bundle $x(p, w)$ is a $\succsim$ maximal bundle within $B(p, w)$.

Of course, if behavior satisfies homogeneity of degree zero and Walras's law, it is still not necessarily rationalizable in any of those senses:

***Example 1:***

Consider the demand function of a consumer who spends all his wealth on the "more expensive" good:

$$x((p_1, p_2), w) = \begin{cases} (0, w/p_2) & \textit{if} \quad p_2 \geq p_1 \\ (w/p_1, 0) & \textit{if} \quad p_2 < p_1 \end{cases}.$$

This demand function is not entirely inconceivable, and yet it is not rationalizable. To see this, assume that it is fully rationalizable or rationalizable by $\succsim$. Consider the two budget sets $B((1, 2), 1)$ and $B((2, 1), 1)$. Since $x((1, 2), 1) = (0, 1/2)$ and $(1/2, 0)$ is an internal bundle in $B((1, 2), 1)$, by any of the two definitions of rationalizability, it must be that $(0, 1/2) \succ (1/2, 0)$. Similarly, $x((2, 1), 1) = (1/2, 0)$ and $(0, 1/2)$ is an internal bundle in $B((2, 1), 1)$. Thus, $(0, 1/2) \prec (1/2, 0)$, a contradiction.

***Example 2:***

A consumer chooses a bundle $(z, z, \ldots, z)$, where $z$ satisfies $z\Sigma p_k = w$.

This behavior is fully rationalized by any preferences according to which the consumer strictly prefers any bundle on the main diagonal over any bundle that is not (because, for example, he cares primarily about purchasing equal quantities from all sellers of the $K$ goods), while on the main diagonal his preferences are according to "the more the better". These preferences rationalize his behavior in the first sense but are not monotonic.

This demand function is also fully rationalized by the monotonic preferences represented by the utility function $u(x_1, \ldots, x_K) = min\{x_1, \ldots, x_K\}$.

***Example 3:***

Consider a consumer who spends $\alpha_k$ of his wealth on commodity $k$ (where $\alpha_k \geq 0$ and $\Sigma_{k=1}^{K} \alpha_k = 1$). This rule of behavior is not formulated as a maximization of some preference relation. It can however be fully rationalized by the preference relation represented by the Cobb-Douglas utility function $u(x) = \Pi_{k=1}^{K} x_k^{\alpha_k}$. A solution $x^*$ to the consumer's problem $B(p, w)$ must satisfy $x_k^* > 0$ for all $k$ (notice that $u(x) = 0$ when $x_k = 0$ for some $k$). Given the differentiability of the preferences, a necessary condition for the optimality of $x^*$ is that $v_k(x^*)/p_k = v_l(x^*)/p_l$ for all $k$ and $l$ where $v_k(x^*) = du/dx_k(x^*) = \alpha_k u(x^*)/x_k^*$ for all $k$. It follows that $p_k x_k^*/p_l x_l^* = \alpha_k/\alpha_l$ for all $k$ and $l$ and thus $x_k^* = \alpha_k w/p_k$ for all $k$.

***Example 4:***

Let $K = 2$. Consider the behavior of a consumer who allocates his wealth between commodities 1 and 2 in the proportion $p_2/p_1$ (the cheaper the good, the higher the share of the wealth devoted to it). Thus, $x_1 p_1/x_2 p_2 = p_2/p_1$ and $x_i(p, w) = (p_j/(p_i + p_j))w/p_i$. This demand function satisfies Walras's law as well as homogeneity of degree zero.

To see that this demand function is fully rationalizable, note that $x_i/x_j = p_j^2/p_i^2$ (for all $i$ and $j$) and thus $p_1/p_2 = \sqrt{x_2}/\sqrt{x_1}$. The quasi-concave function $\sqrt{x_1} + \sqrt{x_2}$ satisfies the condition that the ratio of its partial derivatives is equal to $\sqrt{x_2}/\sqrt{x_1}$. Thus, for any $(p, w)$, the bundle $x(p, w)$ is the solution to the maximization of $\sqrt{x_1} + \sqrt{x_2}$ in $B(p, w)$.

## The Weak and Strong Axioms of Revealed Preferences

We now look for general conditions that will guarantee that a demand function $x(p, w)$ can be fully rationalized (a similar discussion would apply to the other definition of rationalizability that requires that $x(p, w)$ maximizes a monotonic preference relation). Of course, one does not necessarily need these general conditions to determine whether a demand function is rationalizable. Guessing is often an excellent strategy.

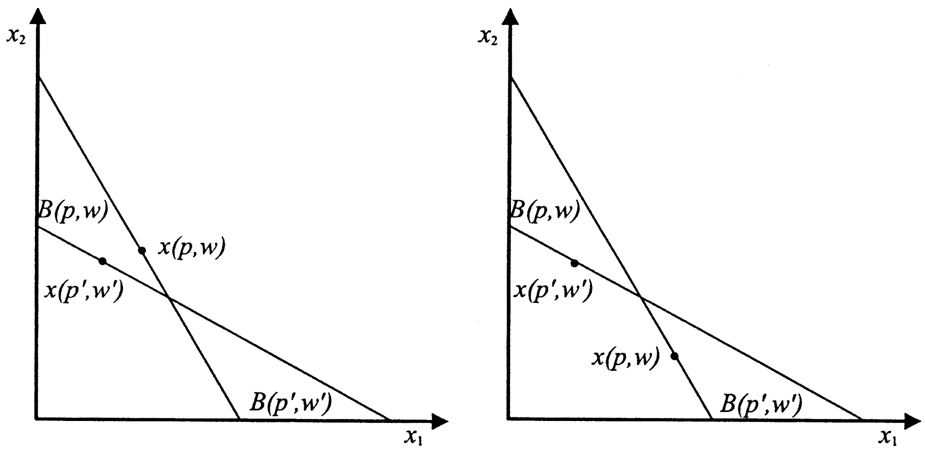



**Figure 5.2**
(a) Satisfies the weak axiom. (b) Does not satisy the weak axiom.

In the general discussion of choice functions, we saw that the weak axiom (WA) was a necessary and sufficient condition for a choice function to be derived from some preference relation. In the proof, we constructed a preference relation out of the choices of the decision maker from sets containing two elements. We showed (by looking into his behavior at the choice set $\{a, b, c\}$) that WA implies that it is impossible for $a$ to be revealed as better than $b$, $b$ revealed as better than $c$, and $c$ revealed as better than $a$. However, in the context of a consumer, finite sets are not within the scope of the choice function.

In the same spirit, adjusting to the context of the consumer, we might try to define $x \succ y$ if there is $(p, w)$ so that both $x$ and $y$ are in $B(p, w)$ and $x = x(p, w)$. In the context of the consumer model the Weak Axiom is read: if $px(p', w') \leq w$ and $x(p, w) \neq x(p', w')$, then $p'x(p, w) > w'$. WA guarantees that it is impossible that both $x \succ y$ and $y \succ x$. However, the defined binary relation is not necessarily complete: there can be two bundles $x$ and $y$ such that for any $B(p, w)$ containing both bundles, $x(p, w)$ is neither $x$ nor $y$. Furthermore, in the general discussion, we guaranteed transitivity by looking at the union of a set in which $a$ was revealed to be better than $b$ and a set in which $b$ was revealed to be as good as $c$. However, when the sets are budget sets, their union is not necessarily a budget set. (See fig. 5.2.)

Apparently WA is not a sufficient condition for extending the binary relation $\succ$, as defined above, into a complete and transitive relation (an example with three goods from Hicks 1956 is discussed in Mas-Colell et al. 1995). A necessary and sufficient condition for a demand function $x$ satisfying Walras's law and homogeneity of degree zero to be rationalized is the following:

### *Strong Axiom of Revealed Preference (SA):*

If $(x^n)_{n=1,\dots,N}$ is a sequence of bundles and $(B(p^n, w^n))_{n=1,\dots,N}$ is a sequence of budget sets so that for all $n \le N-1$, $x^n \neq x^{n+1}$ and $x^n$ is chosen from $B(p^n, w^n)$ which also contains $x^{n+1}$, then $x^1 \notin B(p^N, w^N)$.

The Strong Axiom is basically equivalent to the assumption that the relation $\succ$ derived from revealed behavior is transitive. But $\succ$ is not necessarily a complete relation, and thus we are left with the question of whether $\succ$ can be extended into preferences. Proving that this is possible is beyond the scope of this course. In any case, the SA is "cumbersome," and using it to determine whether a certain demand function is rationalizable may not be a trivial task.

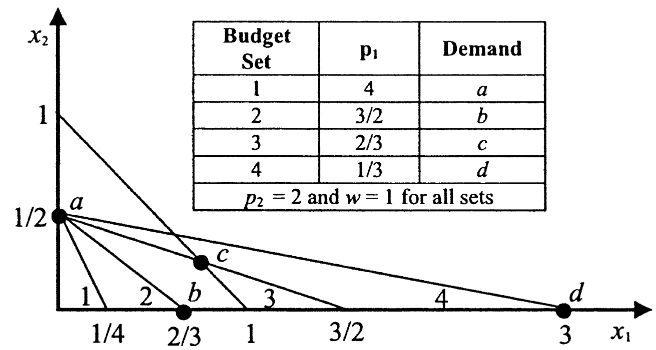
## Decreasing Demand

The consumer model discussed so far constitutes the standard framework for deriving demand. Our intuition tells us that demand for a good falls when its price increases. However, this does not follow from the standard assumptions about the rational consumer's behavior which we have discussed so far. The following is an example of a preference relation that induces demand that is nondecreasing in the price of one of the commodities:

### *An Example in Which Demand for a Good May Increase with Price*

Consider the preferences represented by the following utility function:

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if} \quad x_1 + x_2 < 1 \\ x_1 + 4x_2 & \text{if} \quad x_1 + x_2 \geq 1 \end{cases} .$$




| Budget Set | $p_1$ | Demand |
|---|---|---|
| 1 | 4 | $a$ |
| 2 | 3/2 | $b$ |
| 3 | 2/3 | $c$ |
| 4 | 1/3 | $d$ |
| $p_2 = 2$ and $w = 1$ for all sets | | |

**Figure 5.3**

These preferences might reflect reasoning of the following type: "In the bundle $x$ there are $x_1 + x_2$ units of vitamin A and $x_1 + 4x_2$ units of vitamin B. My first priority is to get enough vitamin $A$. However, once I satisfy my need for 1 unit of vitamin $A$, I move on to my second priority, which is to consume as much as possible of vitamin $B$." (See fig 5.3.)

Consider $x((p_1, 2), 1)$. Changing $p_1$ is like rotating the budget lines around the pivot bundle $(0, 1/2)$. At a high price $p_1$ (as long as $p_1 > 2$), the consumer demands $(0, 1/2)$. If the price is reduced to within the range $2 > p_1 > 1$, the consumer chooses the bundle $(1/p_1, 0)$. So far, the demand for the first commodity indeed increased when its price fell. However, in the range $1 > p_1 > 1/2$ we encounter an anomaly: the consumer buys as much as possible from the second good subject to the "constraint" that the sum of the goods is at least 1, i.e., $x((p_1, 2), 1) = (1/(2 - p_1), (1 - p_1)/(2 - p_1))$.

The above preference relation is monotonic but not continuous. However, we can construct a close continuous preference that leads to demand that is increasing in $p_1$ in a similar domain. Let $\alpha_\delta(t)$ be a continuous and increasing function on $[1 - \delta, 1 + \delta]$ where $\delta > 0$, so that $\alpha_\delta(t) = 0$ for all $t \leq 1 - \delta$ and $\alpha_\delta(t) = 1$ for all $t \geq 1 + \delta$. The utility function

$$u_\delta(x) = (\alpha_\delta(x_1 + x_2)(x_1 + 4x_2)) + (1 - \alpha_\delta(x_1 + x_2)(x_1 + x_2))$$

is continuous and monotonic. For $\delta$ close to 0, the function $u_\delta = u$ except in a narrow area around the set of bundles for which $x_1 + x_2 = 1$.

Now, take two prices, $H > 1$ and $L < 1$, such that a consumer with utility function $u$ consumes more of the first commodity when facing the budget set $((H, 2), 1)$ than when facing the budget set $((L, 2), 1)$ (that is, $1/H > 1/(2 - L)$). When $\delta$ is close enough to 0, the demand induced from $u_\delta$ at $B((H, 2), 1)$ is $(1/H, 0)$. Choose $\epsilon$ such that $1/(2 - L) + \epsilon < 1/H$. For $\delta$ close enough to 0, the bundle in the budget set of $B((L, 2), 1)$ with $x_1 = 1/(2 - L) + \epsilon$ is preferred (according to $u_\delta$) over any other bundle in $B((L, 2), 1)$ with a higher quantity of $x_1$. Thus, for small enough $\delta$, the induced demand for the first commodity at the lower price is at most $1/(2 - L) + \epsilon$, and is thus lower than the demand at the higher price.

## "The Law of Demand"

We are interested in comparing demand in different environments. We have just seen that the classic assumptions about the consumer do not allow us to draw a clear conclusion regarding the relation between a consumer's demand when facing $B(p, w)$ and his demand when facing $B(p + (0, \ldots, \varepsilon, \ldots, 0), w)$.

A clear conclusion can be drawn when we compare the consumer's demand when he faces the budget set $B(p, w)$ to his demand when facing $B(p', x(p, w)p')$. In this comparison we imagine the price vector changing from $p$ to an arbitrary $p'$ and wealth changing in such a way that the consumer has exactly the resources allowing him to consume the same bundle he consumed at $(p, w)$. (See fig. 5.4.)

### *Claim:*

Let $x$ be a demand function satisfying Walras's law and WA. If $w' = p'x(p, w)$, then either $x(p', w') = x(p, w)$ or $[p' - p][x(p', w') - x(p, w)] < 0$.

### *Proof:*

Assume that $x(p', w') \neq x(p, w)$. Then,
$[p' - p][x(p', w') - x(p, w)]$
$= p'x(p', w') - p'x(p, w) - px(p', w') + px(p, w)$
$= w' - w' - px(p', w') + w = w - px(p', w')$

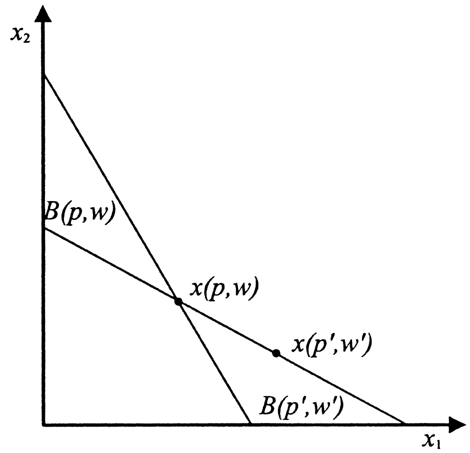



**Figure 5.4**
A compensated price change from $(p, w)$ to $(p', w')$.

(by Walras's law and the assumption that $w' = p'x(p, w)$), and by WA the right-hand side of the equation is less than 0.

## Bibliographic Notes

*Recommended readings*: Kreps 1990, 37–45, Mas-Colell et al. 1995, Chapter 2, A–D, 3, D,J.

The material in this lecture, up to the discussion of differentiability, is fairly standard and closely parallels that found in Arrow and Hahn (1971) and Varian (1984).

# Problem Set 5

**Problem 1.** (*Easy*)
Calculate the demand function for a consumer with the utility function $\sum_k \alpha_k \ln(x_k)$.

**Problem 2.** (*Easy*)
Verify that when preferences are continuous, the demand function $x(p, w)$ is continuous in prices and in wealth (and not only in $p$).

**Problem 3.** (*Easy*)
Show that if a consumer has a homothetic preference relation, then his demand function is homogeneous of degree one in $w$.

**Problem 4.** (*Easy*)
Consider a consumer in a world with $K = 2$, who has a preference relation that is quasi-linear in the first commodity. How does the demand for the first commodity change with $w$?

**Problem 5.** (*Moderately Difficult*)
Let $\succsim$ be a continuous preference relation (not necessarily strictly convex) and $w$ a number. Consider the set $G = \{(p, x) \in \Re^K \times \Re^K |\ x$ is optimal in $B(p, w)\}$. (For some price vectors there could be more than one $(p, x) \in G$.) Calculate $G$ for the case of $K = 2$ and preferences represented by $x_1 + x_2$. Show that (in general) $G$ is a closed set.

**Problem 6.** (*Moderately difficult*)
Determine whether the following behavior patterns are consistent with the consumer model (assume $K = 2$):

a. The consumer's demand function is $x(p, w) = (2w/(2p_1 + p_2), w/(2p_1 + p_2))$.
b. He consumes up to the quantity 1 of commodity 1 and spends his excess wealth on commodity 2.

c. The consumer chooses the bundle $(x_1, x_2)$ which satisfies $x_1/x_2 = p_1/p_2$ and costs $w$. (Does the utility function $u(x) = x_1^2 + x_2^2$ rationalize the consumer's behavior?)

**Problem 7.** (*Moderately difficult*)
In this question, we consider a consumer who behaves differently from the classic consumer we talked about in the lecture. Once again we consider a world with $K$ commodities. The consumer's choice will be from budget sets. The consumer has in mind a preference relation that satisfies continuity, monotonicity, and strict convexity; for simplicity, assume it is represented by a utility function $u$.

The consumer maximizes utility up to utility level $u^0$. If the budget set allows him to obtain this level of utility, he chooses the bundle in the budget set with the highest quantity of commodity 1 subject to the constraint that his utility is at least $u^0$.

a. Formulate the consumer's problem.
b. Show that the consumer's procedure yields a unique bundle.
c. Is this demand procedure rationalizable?
d. Does the demand function satisfy Walras's law?
e. Show that in the domain of $(p, w)$ for which there is a feasible bundle yielding utility of at least $u^0$ the consumer's demand function for commodity 1 is decreasing in $p_1$ and increasing in $w$.
f. Is the demand function continuous?

**Problem 8.** (*Moderately difficult*)
A common practice in economics is to view aggregate demand as being derived from the behavior of a "representative consumer." Give two examples of "well-behaved" consumer preference relations that can induce average behavior that is not consistent with maximization by a "representative consumer." (That is, construct two "consumers," 1 and 2, who choose the bundles $x^1$ and $x^2$ out of the budget set $A$ and the bundles $y^1$ and $y^2$ out of the budget set $B$ so that the choice of the bundle $(x^1 + x^2)/2$ from $A$ and of the bundle $(y^1 + y^2)/2$ from $B$ is inconsistent with the model of the rational consumer.)

LECTURE 6

# Choice over Budget Sets and the Dual Problem

## Indirect Preferences

As an introduction to the first topic in this lecture, let us go back to the general choice function concept discussed in Lecture 3. Having in mind a preference relation $\succsim$ on a set $X$, the decision maker may want to construct a preference relation over the set $D$, the domain of his choice function. When assessing a choice problem in $D$, the decision maker may then ask himself which alternative he would choose if he had to choose from that set. The "rational" decision maker will prefer a set $A$ over a set $B$ if the alternative he intends to choose from $A$ is preferable to that which he intends to choose from $B$. This leads us to the definition of $\succsim^*$, the *indirect preferences* induced from $\succsim$:

$$A \succsim^* B \text{ if } C_{\succsim}(A) \succsim C_{\succsim}(B).$$

The definition of indirect preferences ignores some considerations that might be taken into account when comparing choice sets. Excluded are considerations such as, "I prefer $A - \{b\}$ to $A$ even though I intend to choose $a$ in any case since I am afraid to make a mistake and choose $b$" or "I will choose $a$ from $A$ whether $b$ is available or not. However, since I don't want to have to reject $b$, I prefer $A - \{b\}$ to $A$."

Of course, if $u$ represents $\succsim$ and the choice function is well defined, $v(A) = u(C(A))$ represents $\succsim^*$. We will refer to $v$ as the *indirect utility function*.

Finally, note that sometimes (depending on the set $D$) one can reconstruct the choice function $C_{\succsim}(A)$ from the indirect preferences $\succsim^*$. For example, if $a \in A$ and $A \succ^* A - \{a\}$, then $C_{\succsim}(A) = a$.

### The Consumer's Indirect Utility Function

Let us return to the consumer who chooses bundles from budget sets. He might be interested in formulating indirect preferences when choosing a market to live in or when assessing the effect of tax reforms (which cause changes in prices or wealth) on his welfare. Since a budget set is characterized by the $K+1$ parameters $(p, w)$, the above approach leads to the definition of the indirect preferences $\succsim^*$ on the set $\Re^{K+1}_{++}$ as $(p, w) \succsim^* (p', w')$ if $x(p, w) \succsim x(p', w')$. Interpreting $p$ in the standard manner, as prices prevailing in the market, defining indirect preferences in this way precludes considerations such as, "I prefer to live in an area where alcohol is very expensive even though I drink a lot".

The following are basic properties of the indirect preferences $\succsim^*$, induced from the preferences $\succsim$ on the bundle space. The first is an "invariance to presentation" property, which follows from the definition of indirect preferences independently of the properties of the consumer's preferences. The other three properties depend on the following characteristics of the consumer's preferences: monotonicity (using the partial orderings on the bundle space), continuity (using the topological structure), and convexity (using the algebraic structure).

1. $(\lambda p, \lambda w) \sim^* (p, w)$ (this follows from $x(\lambda p, \lambda w) = x(p, w)$).
2. $\succsim^*$ is nonincreasing in $p_k$ and increasing in $w$ (reducing the scope of the choice is never beneficial, and additional wealth makes it possible to consume bundles containing more of all commodities).
3. If the preference relation $\succsim$ is *continuous*, then so is $\succsim^*$, and there is a continuous function $v$ representing $\succsim^*$ . (The function $x(p, w)$ is continuous. Let $u$ be a continuous function representing $\succsim$; then $u(x(p, w))$ is a continuous utility representation of $\succsim^*$ and thus $\succsim^*$ is continuous.)
4. If $(p^1, w^1) \succsim^* (p^2, w^2)$, then $(p^1, w^1) \succsim^* (\lambda p^1 + (1-\lambda)p^2, \lambda w^1 + (1-\lambda)w^2)$ for all $1 \geq \lambda \geq 0$. (See fig. 6.1.) (Thus, if $v$ represents $\succsim^*$, then it is *quasi-convex*, that is, the set $\{(p, w) | \; v(p, w) \leq v(p^*, w^*)\}$ is convex). To see this, let $z$ be the best bundle in the budget set $B(\lambda p^1 + (1-\lambda)p^2, \lambda w^1 + (1-\lambda)w^2)$. Then $(\lambda p^1 + (1-\lambda)p^2)z \leq \lambda w^1 + (1-\lambda)w^2$ and therefore $p^1 z \leq w^1$ or $p^2 z \leq w^2$. Thus $z \in B(p^1, w^1)$ or $z \in B(p^2, w^2)$ and then $x(p^1, w^1) \succsim z$

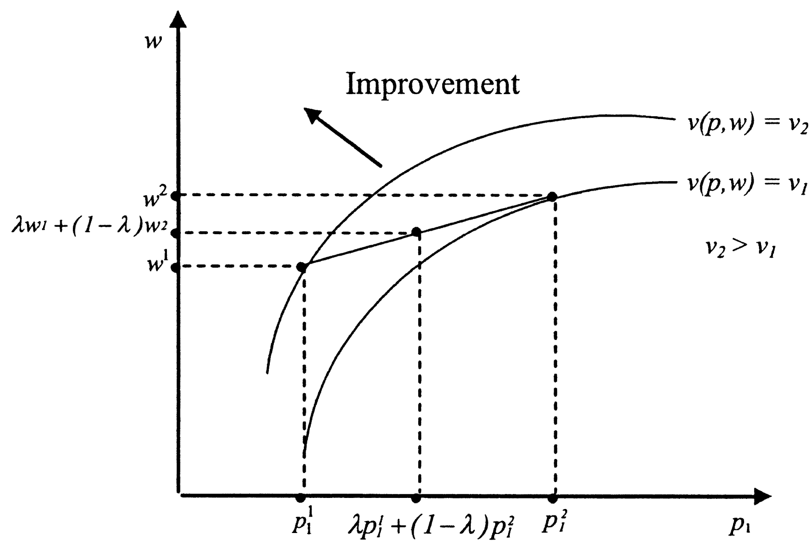



**Figure 6.1**
The indirect utility function is quasi-convex.

or $x(p^2, w^2) \succsim z$. From $x(p^1, w^1) \succsim x(p^2, w^2)$ it follows that $x(p^1, w^1) \succsim z$.

***Example:***

In the single commodity case, each $\succsim^*$-indifference curve is a ray. Assuming monotonicity of $\succsim$, the slope of an indifference curve through $(p_1, w)$ is $x_1(p_1, w) = w/p_1$.

### Roy's Equality

We will now look at a method of deriving the consumer demand function from indirect preferences. The basic idea is that starting from a budget set $(p^*, w^*)$, any change of $\varepsilon$ in the price of commodity $k$ combined with a change of $\varepsilon x_k(p^*, w^*)$ in wealth cannot be undesirable. Thus, when indirect preferences are differentiable, the tangent to the indifference curve of the indirect preferences through $(p^*, w^*)$ gives the demand for that budget set.

***Claim:***

Assume that the demand function satisfies Walras's law. Let $H = \{(p, w) |\ (x(p^*, w^*), -1)(p, w) = 0\}$ for some $(p^*, w^*)$. The hyperplane $H$ is tangent to the $\succsim^*$ indifference curve through $(p^*, w^*)$.

***Proof:***

Of course $(p^*, w^*) \in H$. For any $(p, w) \in H$, the bundle $x(p^*, w^*) \in B(p, w)$. Hence $x(p, w) \succsim x(p^*, w^*)$, and thus $(p, w) \succsim^* (p^*, w^*)$.

In the case in which $\succsim^*$ is represented by differentiable $v$,

$$H = \{(p, w) |\ (\partial v/\partial p_1(p^*, w^*), \dots, \partial v/\partial p_K(p^*, w^*), \partial v/\partial w(p^*, w^*))(p - p^*, w - w^*) = 0\}.$$

From the above claim and since $w^* = p^* x(p^*, w^*)$ we have also

$$H = \{(p, w) |\ (x(p^*, w^*), -1)(p - p^*, w - w^*) = 0\}.$$

Therefore, the vector

$$(\partial v/\partial p_1(p^*, w^*), \dots, \partial v/\partial p_K(p^*, w^*), \partial v/\partial w(p^*, w^*))$$

is proportional to the vector

$$(x_1(p^*, w^*), \dots, x_K(p^*, w^*), -1),$$

and thus, $-[\partial v/\partial p_k(p^*, w^*)]/[\partial v/\partial w(p^*, w^*)] = x_k(p^*, w^*)$.

## Dual Problems

In normal discourse, we consider the following two statements to be equivalent:

1. The maximal distance a turtle can travel in 1 day is 1 km.
2. The minimal time it takes a turtle to travel 1 km is 1 day.

This equivalence actually relies on two "hidden" assumptions:

a. For (1) to imply (2) we need to assume the turtle travels a positive distance in any period of time. Contrast this with the case in which the turtle's speed is 2 km/day but, after half a day, it must rest for half a day. Then, the maximal distance it can travel in 1 day is 1 km but it can travel this distance in only half a day.

b. For (2) to imply (1) we need to assume the turtle cannot "jump" a positive distance in zero time. Contrast this with the case in which the turtle's speed is 1 km/day but after a day of traveling it can "jump" 1 km. Thus, it needs 1 day to travel 1 km but within 1 day it can travel 2 km.

The assumptions that in any positive interval of time the turtle can travel a positive distance and that the turtle cannot "jump" are sufficient for the equivalence of (1) and (2). Let $M(t)$ be the maximal distance the turtle can travel in time $t$. Assume that the function $M$ is strictly increasing and continuous. Then, the statement, "The maximal distance a turtle can travel in $t^*$ is $x^*$" is equivalent to the statement, "The minimal time it takes a turtle to travel $x^*$ is $t^*$."

If the maximal distance that the turtle can pass within $t^*$ is $x^*$, and if it covers the distance $x^*$ in $t < t^*$ then, by the strict monotonicity of $M$, the turtle would cover a distance larger than $x^*$ in $t^*$, a contradiction.

If it takes $t^*$ for to the turtle to cover the distance $x^*$ and if it passes the distance $x > x^*$ in $t^*$, then by the continuity of $M$ at some $t < t^*$ the turtle will already be beyond the distance $x^*$, a contradiction.

## The Dual Consumer Problem

Let $u$ be a utility function that is continuous and monotonic. Applying the duality idea to the consumer problem, we compare the following pair of maximization problems:

### *The prime problem* $P(p, w^*)$

Find a bundle maximizing utility given an expense level $w^*$, that is,

$$max_x\{u(x)| \;\; px \leq w^*\}.$$

• • •

***The dual problem $D(p, u^*)$***

Find a bundle minimizing the expenses needed to obtain a level of utility $u^*$, that is,

$$min_x\{px|\ u(x) \geq u^*\}.$$

***Claim:***

1. If $x^*$ is the solution to the problem $P(p, w^*)$, then it is also the solution to the dual problem $D(p, u(x^*))$.
2. If $x^*$ is a solution to the dual problem $D(p, u^*)$, then it is also the solution to the problem $P(p, px^*)$.

***Proof:***

1. If $x^*$ is not a solution to the dual problem $D(p, u(x^*))$, then there exists a strictly cheaper bundle $x$ for which $u(x) \geq u(x^*)$. For some positive vector $\varepsilon$ (that is, $\varepsilon_k > 0$ for all $k$), it still holds that $p(x+\varepsilon) < px^* \leq w$. By monotonicity $u(x+\varepsilon) > u(x) \geq u(x^*)$, contradicting the assumption that $x^*$ is a solution to $P(p, w^*)$.
2. If $x^*$ is not a solution to the problem $P(p, w^*)$, then there exists an $x$ such that $px \leq px^*$ and $u(x) > u(x^*) \geq u^*$. By continuity, for some nonnegative vector $\varepsilon \neq 0$, $x - \varepsilon$ is a bundle such that $u(x-\varepsilon) > u^*$ and $p(x-\varepsilon) < px^*$, contradicting the assumption that $x^*$ is a solution to $D(p, u^*)$.

## The Hicksian Demand Function

Assume that the dual problem $D(p, u)$ has a unique solution. This is the case, for example, if $u$ represents strictly convex continuous preferences. The Hicksian demand function $h(p, u)$ is the solution to $D(p, u)$. This concept is analogous to the demand function in the prime problem.

Here are some properties of the Hicksian demand function:

1. $h(\lambda p, u) = h(p, u)$. If $x$ is a solution to the problem $D(p, u)$, it is also a solution to the problem $D(\lambda p, u)$. The function $\lambda px$ is a positive linear transformation of $px$; thus, the problem $min_x\{\lambda px|\ u(x) \geq u\}$ has the same solution as the problem $min_x\{px|\ u(x) \geq u\}$.
2. $h_k(p, u)$ is nonincreasing in $p_k$. Note that for every $p'$ $ph(p, u) \leq ph(p', u)$ since $h(p', u)$ also satisfies the constraint of achieving a utility level of at least $u$ and $h(p, u)$ is the cheapest bundle satisfying the constraint. Similarly, $p'h(p', u) \leq p'h(p, u)$. Thus,
$$(p - p')(h(p, u) - h(p', u)) = p[h(p, u) - h(p', u)] \\ + p'[h(p', u) - h(p, u)] \leq 0.$$
When $p - p' = (0, \ldots, \epsilon, \ldots, 0)$ we get that $h_k(p, u) - h_k(p', u) \leq 0$. Thus, increasing the price of commodity $k$ has a nonpositive effect on Hicksian demand.
3. $h(p, u)$ is continuous in $p$ (verify!).

Define $e(p, u) = ph(p, u)$ to be the *expenditure function*. This concept is analogous to the indirect utility function in the prime problem. Here are some properties of the expenditure function:

1. $e(\lambda p, u) = \lambda e(p, u)$ (it follows from $h(\lambda p, u) = h(p, u)$).
2. $e(p, u)$ is non-decreasing in $p_k$ and strictly increasing in $u$.
3. $e(p, u)$ is continuous in $p$ (this follows from the continuity of $h(p, u)$).
4. $e(p, u)$ is concave in $p$ (not only in $p_k$). To prove this, let $x = h(\lambda p^1 + (1 - \lambda)p^2, u^*)$. Since $u(x) = u^*$, $e(p^i, u^*) \leq p^i x$; thus $e(\lambda p^1 + (1 - \lambda)p^2, u^*) = (\lambda p^1 + (1 - \lambda)p^2)x \geq \lambda e(p^1, u^*)$ $+ (1-\lambda)e(p^2, u^*)$.

***Claim (the Dual Roy's Equality):***

The hyperplane $H = \{(p, e)|\ e = ph(p^*, u^*)\}$ is tangent to the graph of the function $e = e(p, u^*)$ at point $p^*$.

***Proof:***

Since $ph(p^*, u^*) \geq ph(p, u^*)$ for all price vectors $p$, the hyperplane $H$ lies on one side of the graph of the function $e = ph(p, u^*)$ and intersects the graph at the point $(p^*, e(p^*, u^*))$.

## Bibliographic Notes

*Recommended readings*: Kreps 1990, 45–63; MWG, chapter 2, E–F; 3, D–G, I–J.

Roy and Hicks are the sources for most of the material in this lecture. Specifically, the concept of the indirect utility function is due to Roy (1942); the concept of the expenditure function is due to Hicks (1946); and the concepts of consumer surplus used in Problem 6 are due to Hicks (1939). See also McKenzie (1957). For a full representation of the duality idea, see, for example, Varian (1984) and Diewert (1982).

# Problem Set 6

**Problem 1.** (*Easy*)
In a world with two commodities, consider a consumer's preferences that are represented by the utility function $u(x_1, x_2) = min\{x_1, x_2\}$.

a. Calculate the consumer's demand function.
b. Verify that the preferences satisfy convexity.
c. Calculate the indirect utility function $v(p, w)$.
d. Verify Roy's Equality.
e. Calculate the expenditure function $e(p, u)$ and verify Dual Roy's Equality.

**Problem 2.** (*Moderate*)
Imagine that you are reading a paper in which the author uses the indirect utility function $v(p_1, p_2, w) = w/p_1 + w/p_2$. You suspect that the author's conclusions in the paper are the outcome of the "fact" that the function $v$ is inconsistent with the model of the rational consumer. Take the following steps to make sure that this is not the case:

a. Use Roy's Equality to derive the demand function.
b. Show that if demand is derived from a smooth utility function, then the indifference curve at the point $(x_1, x_2)$ has the slope $-\sqrt{x_2}/\sqrt{x_1}$.
c. Construct a utility function with the property that the ratio of the partial derivatives at the bundle $(x_1, x_2)$ is $\sqrt{x_2}/\sqrt{x_1}$.
d. Calculate the indirect utility function derived from this utility function. Do you arrive at the original $v(p_1, p_2, w)$? If not, can the original indirect utility function still be derived from another utility function satisfying the property in (c).

**Problem 3.** (*Moderate*)
A consumer with wealth $w$ is interested in purchasing only one unit of one of the items included in a (finite) set $A$. All items are *indivisible*. The consumer does not derive any "utility" from leftover wealth. The consumer evaluates commodity $x \in A$ by the number $V_x$ (where the value of not purchasing any of the goods is 0). The price of commodity $x \in A$ is $p_x > 0$.

a. Formulate the consumer's problem.

b. Check the properties of the indirect utility function (homogeneity of degree zero, monotonicity, continuity and quasi-convexity).
c. Calculate the indirect utility function for the case in which $A = \{a, b\}$ and $V_a > V_b > 0$.

**Problem 4.** (*Moderate*)
Show that if the utility function is continuous, then so is the Hicksian demand function $h(p, u)$.

**Problem 5.** (*Moderate*)
A commodity $k$ is *Giffen* if the demand for the $k$-th good, $x_k(p, w)$, is increasing in $p_k$. A commodity $k$ is *inferior* if the demand for the commodity decreases with wealth. Show that if a commodity $k$ is Giffen in some neighborhood of $(p, w)$, then $k$ is inferior.

**Problem 6.** (*Moderate*)
One way to compare budget sets is by using the relation $\succsim^*$ as defined in the text. According to this approach, the comparison between $(p, w)$ and $(p', w)$ is made by comparing two numbers $u(x(p, w))$ and $u(x(p', w))$, where $u$ is a utility function defined on the space of the bundles.

Following are two other approaches for making such comparisons using "concrete terms."

Define:

$$CV(p, p', w) = w - e(p', u) = e(p, u) - e(p', u)$$

where $u = u(x(p, w))$.

This is the answer to the question: What is the change in wealth that would be equivalent, from the perspective of $(p, w)$, to the change in price vectors from $p$ to $p'$?

Define:

$$EV(p, p', w) = e(p, u') - w = e(p, u') - e(p', u')$$

where $u' = u(x(p', w))$.

This is the answer to the question: What is the change in wealth that would be equivalent, from the perspective of $(p', w)$, to the change in price vectors from $p$ to $p'$?

Now, answer the following questions regarding a consumer in a two-commodity world with a utility function $u$:

a. For the case $u(x_1, x_2) = x_1 + x_2$, calculate the two "consumer surplus" measures.

b. Explain why the two measures may give different values for some other utility functions.
c. Explain why the two measures are identical if the individual has quasi-linear preferences in the second commodity and in a domain where the two commodities are consumed in positive quantities.
d. Assume that the price of the second commodity is fixed and that the price vectors differ only in the price of the first commodity. What is the relation of the two measures to the "area below the demand function" (which is a standard third definition of consumer surplus)?

LECTURE 7

# Production

## The Producer: The Basic Model

We will turn now to a very brief discussion of the basic concepts in classic producer theory. As this involves only a few new abstract ideas, we make do with a short introduction to the concepts and implicit assumptions.

Usually we view the firm as a special type of rational decision maker. Recall that when discussing the consumer we imposed a strong structure on the choice sets but few constraints on the preferences. In contrast, classic producer theory assigns the producer a highly structured target function but fewer constraints on the choice sets.

Let $1, \ldots, K$ be commodities. The producer's choice will be made from subsets of the "grand set," which will be taken to be a $K$-dimensional Euclidean space. A vector $z$ in this space is interpreted as a production combination; positive components in $z$ are interpreted as outputs and negative components as inputs.

## Producer's Preferences

It is assumed that the goal of the *producer (firm)* is to maximize profits. The competitive producer faces a vector of prices $p = (p_k)_{k=1,\ldots,K}$ (for inputs and outputs). If he chooses $z$, his profits (revenues minus costs) will be $pz = \sum_{k=1}^{k} p_k \cdot z_k$. In other words, it is assumed that his preferences over any set of possible production combinations are represented by the utility function $pz$.

## Technology

A producer's choice set is called a *technology* and specifies the production constraints.

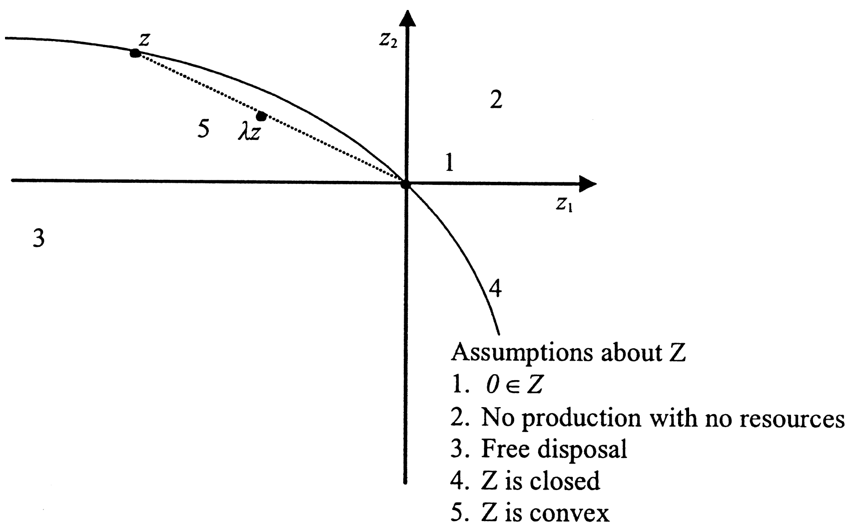



**Figure 7.1**

The following restrictions are usually placed on the technology space (fig. 7.1):

1. $0 \in Z$ (this is interpreted to mean that the producer can remain "idle").
2. There is no $z \in Z \cap \Re^K_+$ besides the vector 0 (no production with no resources).
3. *Free disposal*: if $z \in Z$ and $z' \leq z$, then $z' \in Z$ (nothing prevents the producer from being inefficient in the sense that it uses more resources than necessary to produce the same amount of commodities).
4. $Z$ is a closed set.
5. $Z$ is a convex set. (This assumption embodies decreasing marginal productivity. Together with the assumption that $0 \in Z$, it implies *nonincreasing returns to scale*: if $z \in Z$, then for every $\lambda < 1$, $\lambda z \in Z$.)

## The Production Function

Consider the case in which commodity $K$ is produced from commodities $1, \ldots, K-1$, that is, for all $z \in Z$, $z_K \geq 0$ and for all $k \neq K$, $z_k \leq 0$. In this case, another intuitive way of specifying the technological constraints on the producer is by a *production function* which specifies, for any positive vector of inputs $v \in R_+^{K-1}$, the maximum amount of commodity $K$ that can be produced.

If we start from technology $Z$, we can derive the production function by defining

$$f(v) = max\{x \mid (-v, x) \in Z\}.$$

Alternatively, if we start from the production function $f$, we can derive the "technology" by defining $Z(f) = \{(-w, x) \mid x \leq y$ and $w \geq v$ for some $y = f(v)\}$. If the function $f$ satisfies the assumptions of $f(0) = 0$, continuity, and concavity, then $Z(f)$ satisfies the above assumptions.

## The Supply Function

We will now discuss the producer's behavior. The producer's problem is defined as $max_{z \in Z} pz$.

The existence of a unique solution for the producer problem requires some additional assumptions such as that $Z$ be *bounded from above* (that is, there is some bound $B$ such that $B \geq z_k$ for any $z \in Z$) and that $Z$ be *strictly convex* (that is, if $z$ and $z'$ are in $Z$, then the combination $\lambda z + (1 - \lambda)z'$ is an internal point in $Z$ for any $1 > \lambda > 0$).

When the producer's problem has a unique solution, we denote it by $z(p)$. We refer to the function $z(p)$ as the *supply function*. Note that it specifies both the producer's supply of outputs and its demand for inputs. The *profit function* $\pi(p) = max_{z \in Z} pz$ is analogous to the indirect utility function in the consumer model.

Recall that when discussing the consumer, we specified the preferences and we described his behavior as making a choice from a budget set that had been determined by prices. The consumer's behavior (demand) specified the dependence of his consumption on prices. In the case of the producer, we specify the technology and we describe his behavior as maximizing a profit function which is determined by prices. The producer's behavior (supply) specifies the dependence of output and the consumption of inputs on prices.

In the case of the producer, preferences are linear and the constraint is a convex set, whereas in the consumer model the constraint is a linear inequality and preferences are convex. The structure (continuity and convexity) is imposed on the producer's choice set

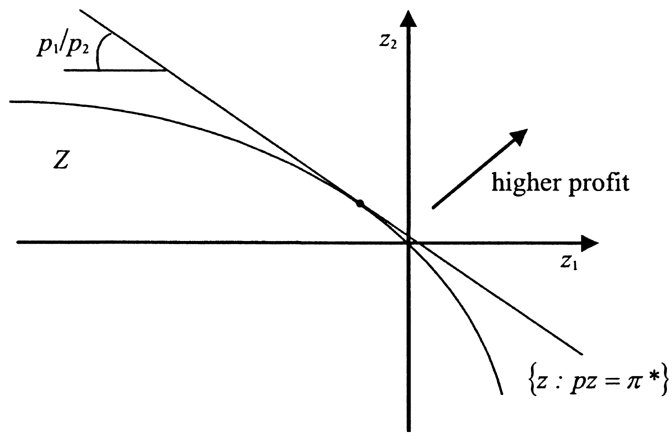



**Figure 7.2**
Profit maximization.

and on the consumer's preferences. Thus, the producer's problem is similar to that of the consumer's *dual* problem. (See fig. 7.2.)

## Properties of the Supply and Profit Functions

Let us turn to some of the properties of the supply and profit functions. The properties and their proofs are analogous to the properties and proofs in the discussion of the consumer's dual problem.

### *Supply Function*

1. $z(\lambda p) = z(p)$. (The producer's preferences are induced by the price vector $p$ and are identical to those induced by the price vector $\lambda p$.)
2. $z$ is continuous.
3. Assume the supply function is well defined. If $z(p) \neq z(p')$, we have $(p - p')[z(p) - z(p')] = p[z(p) - z(p')] + p'[z(p') - z(p)] > 0$. In particular, if (only) the $k$th price increases, $z_k$ increases; that is, if $k$ is an output ($z_k > 0$), the supply of $k$ increases; and if $k$ is an input ($z_k < 0$), the demand for $k$ decreases. Note that this result, called *the law of supply*, applies to the standard supply

function (unlike the law of demand, which was applied to the compensated demand function).

### *Profit Function*

1. $\pi(\lambda p) = \lambda\pi(p)$. (Follows from $z(\lambda p) = z(p)$.)
2. $\pi$ is continuous. (Follows from the continuity of the supply function.)
3. $\pi$ is convex. (For all $p$,$p'$ and $\lambda$, if $z$ maximizes profits with $\lambda p + (1-\lambda)p'$ then $\pi(\lambda p + (1-\lambda)p') = \lambda pz + (1-\lambda)p'z \leq \lambda\pi(p) + (1-\lambda)\pi(p')$.)
4. *Hotelling's lemma*: For any vector $p^*$, $\pi(p) \geq pz(p^*)$ for all $p$. Therefore, the hyperplane $\{(p,\pi) \mid \pi = pz(p^*)\}$ is tangent to $\{(p,\pi) \mid \pi = \pi(p)\}$, the graph of function $\pi$ at the point ($p^*$, $\pi(p^*)$). If $\pi$ is differentiable, then $d\pi/dp_k(p^*) = z_k(p^*)$.
5. From Hotelling's lemma it follows that if $\pi$ is differentiable, then $dz_j/dp_k(p^*) = dz_k/dp_j(p^*)$.

## The Cost Function

If we are only interested in the firm's behavior in the output market (but not in the input markets), it is sufficient to specify the costs associated with the production of any combination of outputs as opposed to the details of the production function. Thus, for a producer of the commodities $L+1,\ldots,K$, we define $c(p,y)$ to be the minimal cost associated with the production of the combination $y \in \Re_+^{K-L}$ given the price vector $p \in \Re_{++}^L$ of the input commodities $1,\ldots,L$. If the model's primitive is a technology $Z$, we have $c(p,y) = min_a\{pa|\ (-a,y) \in Z\}$. (See fig. 7.3.)

## Discussion

In the conventional economic approach we allow the consumer "general" preferences but restrict producer goals to profit maximization. Thus, a consumer who consumes commodities in order to destroy his health is within the scope of our discussion, while a producer who cares about the welfare of his workers or has in mind a target other than profit maximization is not. This is odd since there

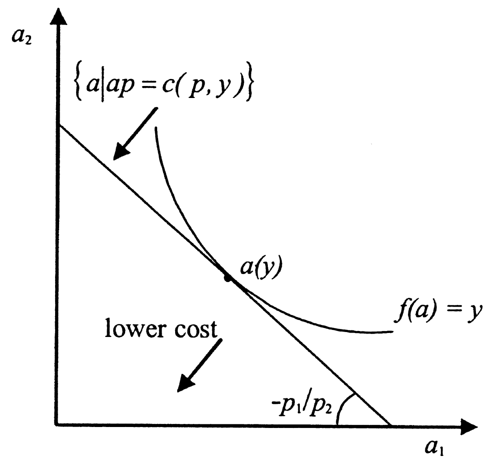



**Figure 7.3**
Cost Minimization.

are various empirically plausible alternative targets for a producer. For example, it seems that the goal of some producers is to increase production subject to not incurring a loss. Some firms are managed so as to increase the managers' salaries with less regard for the level of profits.

I sometimes wonder why this difference exists between the generality of consumer preferences and the narrowness of the producer objectives. It might be that this is simply the result of mathematical convenience. I don't think this is a result of an ideological conspiracy. But, by making profit maximization the key assumption about producer behavior, do we not run the risk that students will interpret it to be the exclusive normative criterion guiding a firm's actions?

## Bibliographic Notes

*Recommended readings*: Kreps 1990, Chapter 8; Mas-Colell et al. 1995, Chapter 5, A–D,G.

The material in this lecture (apart from the discussion) is standard and can be found in any microeconomics textbook. Debreu (1959) is an excellent source.

# Problem Set 7

**Problem 1.** (*Easy*)
Assume that technology $Z$ and the production function $f$ describe the same producer who produces commodity $K$ using inputs $1, \dots, K-1$. Show that $Z$ is a convex set if and only if $f$ is a concave function.

**Problem 2.** (*Boring*)
Here is a very standard exercise (if you have not done it in the past, it may be "fun" to do it "once in a lifetime"): Calculate the supply function $z(p)$ and the profit function $\pi(p)$ for each of the following production functions:

- $f(a) = a_1^\alpha$ for $\alpha \leq 1$.
- $g(a) = \alpha a_1 + \beta a_2$ for $\alpha > 0$ and $\beta > 0$.
- $h(a) = min\{a_1, a_2\}$.
- $i(a) = (a_1^\alpha + a_2^\alpha)^{1/\alpha}$ for $\alpha \leq 1$.

**Problem 3.** (*Easy*)
Consider a producer who uses $L$ inputs to produce $K-L$ outputs. Show the following:

- $C(\lambda w, y) = \lambda C(w, y)$.
- $C$ is nondecreasing in any input price $w_k$.
- $C$ is concave in $w$.
- Shepherd's lemma: If $C$ is differentiable, $dC/dw_k(w, y) = a_k(w, y)$ (the $k$th input commodity).
- If $C$ is twice continuously differentiable, then for any two commodities $j$ and $k$ $da_k/dw_j(w, y) = da_j/dw_k(w, y)$.

**Problem 4.** (*Moderately difficult.* Based on Radner (1993).)
It is usually assumed that the cost function $C$ is convex in the output vector. Much of the research on production has been aimed at investigating whether the convexity assumptions can be induced in more detailed models. Convexity often fails when the product is related to the gathering of information or data processing.

Consider, for example, a firm conducting a telephone survey immediately following a TV program. Its goal is to collect information about as many viewers as possible within 4 units of time. The wage paid to each worker is $w$ (even when he is idle). In one unit of time, a worker can talk to one respondent or be involved in the transfer of information to or from exactly one colleague. At the end of the 4 units of time, the collected information must be in the hands of one colleague (who will announce the results).

- What is the firm's product?
- Calculate the cost function and examine its convexity.

**Problem 5.** (*Moderately difficult*)
Consider a firm producing one commodity using $L$ inputs, which maximizes production subject to the constraint of nonnegative profits. Show some interesting properties of such a firm's behavior.

**Problem 6.** (*Standard*)
An event that could have happened with probability 0.5 either did or did not occur. A firm has to provide a report of the form "the event occurred" or "the event did not occur." The quality of the report, $q$ (the firm's product), is the probability that the report is correct. The firm employs $k$ experts (input) to prepare the report. Each of them receives an independent signal whether the event occurred or not, which is correct with probability $1 > p > 0.5$.

- Calculate the production function $q = f(k)$ for (at least) $k = 1, 2, 3, \ldots$.
- We say that a "discrete" production function is concave if the sequence of marginal product is nonincreasing. Is the firm's production function concave?
  Assume that the firm needs information in order to make a decision whether to invest amount $m$ that will yield revenue $\alpha m$ if the event occurs, and 0 otherwise; the decision maker chooses $k$ in order to maximize expected profits. Assume that the wage of each worker is $w$.
- Explain why it is true that if $f$ is concave, the firm chooses $k^*$ so that the $k^*$th worker is the last one for whom marginal revenue exceeds the cost of a single worker.
- Is this conclusion true in our case?

LECTURE 8

# Expected Utility

## Lotteries

When thinking about decision making, we often distinguish between actions and consequences. An action is chosen and leads to a consequence. The rational man has preferences over these consequences and is meant to choose a feasible action that leads to the most desired consequence. In our discussion of the rational man, we have so far not distinguished between actions and consequences since it was unnecessary for modeling situations where each action deterministically leads to a particular consequence.

In this lecture we will discuss a decision maker in an environment in which the correspondence between actions and consequences is not deterministic but *stochastic*. The choice of an action is viewed as choosing a "lottery ticket" where the prizes are the consequences. We will be interested in preferences and choices over the set of lotteries.

Let $Z$ be a set of consequences (prizes). In this lecture we assume that $Z$ is a finite set. A *lottery* is a probability measure on $Z$, i.e., a lottery $p$ is a function that assigns a nonnegative number $p(z)$ to each prize $z$, where $\Sigma_{z \in Z} p(z) = 1$. The number $p(z)$ is taken to be the objective probability of obtaining the prize $z$ given the lottery $p$.

Denote by $[z]$ the degenerate lottery for which $[z](z) = 1$. We will use the notation $\alpha x \oplus (1-\alpha)y$ to denote the lottery in which the prize $x$ is realized with probability $\alpha$ and the prize $y$ with probability $1-\alpha$.

Denote by $L(Z)$ the (infinite) space containing all lotteries with prizes in $Z$. Given the set of consequences $Z$, the space of lotteries $L(Z)$ can be identified with a simplex in Euclidean space: $\{x \in \Re^Z_+ \mid \Sigma x_z = 1\}$. Recall that $\Re^Z_+$ is the set of functions from $Z$ into $\Re_+$. The extreme points of the simplex correspond to the degenerate lotteries.

We will talk about preferences over $L(Z)$. An implicit assumption is that the decision maker does not care about the nature of the random factors but only about the distribution of consequences. To appreciate this point, consider a case in which the probability of rain is $1/2$ and $Z = \{z_1, z_2\}$, where $z_1$ = "having an umbrella" and $z_2$ = "not having an umbrella." A "lottery" in which you have $z_1$ if it is raining and $z_2$ if it is not is *not* equivalent to the "lottery" in which you have $z_1$ if it is not raining and $z_2$ if it is. Thus, we have to be careful not to apply the model in contexts where the attitude toward the consequence depends on the event realized in each possible contingence.

## Preferences

Let us start by thinking about examples of "sound" preferences over a space $L(Z)$. Following are some examples:

- *Preference for uniformity*: The decision maker prefers the lottery that is less disperse where dispersion is measured by $\Sigma_k (p_k - 1/K)^2$.
- *Extreme preference for certainty*: The decision maker prefers $p$ to $q$ if $max_z p(z)$ is greater than $max_z q(z)$.
- *The size of the support*: The decision maker evaluates each lottery by the number of prizes that can be realized with positive probability, that is, by the size of the support of the lottery, $supp(p) = \{z | p(z) > 0\}$. He prefers the one with the smaller support ($p \succsim q$ if $|supp(p)| \leq |supp(q)|$).

In the examples above, the preferences ignored the consequences and were dependant on the probability vectors alone. In the following examples, the preferences take into account the evaluation of the prizes as well.

- *Increasing the probability of a "good" outcome*: The set $Z$ is partitioned into two disjoint sets $G$ and $B$ (good and bad), and the decision maker prefers the lottery that yields "good" prizes with higher probability.
- *The worst case*: The decision maker evaluates lotteries by the *worst possible* case. He attaches a number $v(z)$ to each prize $z$ and $p \succsim q$ if $min\{v(z)|\ p(z) > 0\} \geq min\{v(z)|\ q(z) > 0\}$. This criterion is often used in Computer Science, where one algorithm

is preferred to another if it functions better in the worst case independently of the likelihood of the worst case occurring.

- *Comparing the most likely prize*: The decision maker considers the prize in each lottery which is most likely (breaking ties in some arbitrary way) and compares two lotteries according to a basic preference relation over $Z$.
- *Lexicographic preferences*: The prizes are ordered $z_1, \dots, z_K$ and the lottery $p$ is preferred to $q$ if $(p(z_1), \dots, p(z_K)) \geq_L (q(z_1), \dots, q(z_K))$.
- *Expected utility*: A number $v(z)$ is attached to each prize and a lottery is evaluated according to its expected $v$, that is, according to $\Sigma_z p(z)v(z)$. Thus,

$$p \succsim q \text{ if } U(p) = \Sigma_{z\in Z} p(z)v(z) \geq U(q) = \Sigma_{z\in Z} q(z)v(z).$$

The richness of examples calls for the classification of preference relations over lotteries and the study of properties that these relations satisfy. The methodology we follow is to formally state general principles (axioms) that may apply to preferences over the space of lotteries. Each axiom carries with it a consistency requirement or involves a procedural aspect of decision making. The axiomatization of a family of preference relations provides justification for focusing on that specific family.

## Von Neumann-Morgenstern Axiomatization

The version of the von Neumann-Morgenstern axiomatization presented here uses two axioms, the independence and continuity axioms.

### *The Independence Axiom*

In order to state the first axiom we require an additional concept, called *Compound lotteries* (fig. 8.1): Given a $K$-tuple of lotteries $(p^k)$ and a $K$-tuple of nonnegative numbers $(\alpha_k)_{k=1,\dots,K}$ that sum up to 1, define $\oplus_{k=1}^K \alpha_k p^k$ to be the lottery for which $(\oplus_{k=1}^K \alpha_k p^k)(z) = \Sigma_{k=1}^K \alpha_k p^k(z)$. Verify that $\oplus_{k=1}^K \alpha_k p^k$ is indeed a lottery. When only two lotteries $p^1$ and $p^2$ are involved, we use the notation $\alpha_1 p^1 \oplus (1 - \alpha_1)p^2$.

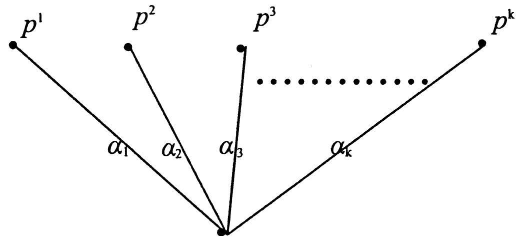



**Figure 8.1**
The compound lottery $\oplus_{k=1}^{K}\alpha_k p^k$.

We think of $\oplus_{k=1}^{K}\alpha_k p^k$ as a compound lottery with the following two stages:

*Stage 1*: It is randomly determined which of the lotteries $p^1, \ldots, p^K$ is realized; $\alpha_k$ is the probability that $p^k$ is realized.
*Stage 2*: The prize finally received is randomly drawn from the lottery determined in stage 1.

When we compare two compound lotteries, $\alpha p \oplus (1-\alpha)r$ and $\alpha q \oplus (1-\alpha)r$, we tend to simplify the comparison and form our preference on the basis of the comparison between $p$ and $q$. This intuition is translated into the following axiom:

***Independence (I):***

For any $p, q, r \in L(Z)$ and any $\alpha \in (0, 1)$,

$$p \succsim q \text{ iff } \alpha p \oplus (1-\alpha)r \succsim \alpha q \oplus (1-\alpha)r.$$

The following property follows from $I$ :

***I*:***

$\oplus_{k=1}^{K}\alpha_k p^k \succsim \oplus_{k=1}^{K}\alpha_k q^k$ when $p^k = q^k$ for all $k$ but $k^*$ iff $p^{k^*} \succsim q^{k^*}$.

To see it,

$$\oplus_{k=1,\ldots,K}\alpha_k p^k = \alpha_{k^*}p^{k^*} \oplus (1-\alpha_{k^*})(\oplus_{k \neq k^*}[\alpha_k/(1-\alpha_{k^*})]p^k) \succsim \alpha_{k^*}q^{k^*} \oplus (1-\alpha_{k^*})(\oplus_{k \neq k^*}[\alpha_k/(1-\alpha_{k^*})]q^k) = \oplus_{k=1}^{K}\alpha_k q^k \text{ iff } p^{k^*} \succsim q^{k^*}.$$

***The Continuity Axiom***

Once again we will employ a continuity assumption that is basically the same as the one we employed for the consumer model. Continuity means that the preferences are not overly sensitive to small changes in the probabilities.

***Continuity (C):***

If $p \succ q$, then there are neighborhoods $B(p)$ of $p$ and $B(q)$ of $q$ (when presented as vectors in $R^{|Z|}$), such that

$$\text{for all } p' \in B(p) \text{ and } q' \in B(q), p' \succ q'.$$

The continuity assumption implies (verify!) the following property that is sometimes presented as an alternative definition of continuity:

***C*:***

If $p \succ q \succ r$, then there exists $\alpha \in (0, 1)$ such that

$$q \sim [\alpha p \oplus (1 - \alpha)r].$$

Let us check whether some of the examples we discussed earlier satisfy these two axioms.

- *Expected utility*: Note that the function $U(p)$ is linear:

$$U(\oplus_{k=1}^{K} \alpha_k p^k) = \sum_{z \in Z} [\oplus_{k=1}^{K} \alpha_k p^k](z) v(z) = \sum_{z \in Z} [\sum_{k=1}^{K} \alpha_k p^k(z)] v(z)$$
$$= \sum_{k=1}^{K} \alpha_k [\sum_{z \in Z} p^k(z) v(z)] = \sum_{k=1}^{K} \alpha_k U(p^k).$$

  It follows that any such preference relation satisfies $I$. Since the function $U(p)$ is continuous in the probability vector, it also satisfies $C$.
- *Increasing the probability of a "good" consequence*: Such a preference relation satisfies the two axioms since it can be represented by the expectation of $v$ where $v(z) = 1$ for $z \in G$ and $v(z) = 0$ for $z \in B$.

- *Extreme preference for certainty*: This preference relation is continuous (as the function $max\{p_1, \ldots, p_K\}$ which represents it is continuous in probabilities). It does not satisfy $I$ since, for example, although $[z_1] \sim [z_2]$, $1/2[z_1] \oplus 1/2[z_1] \succ 1/2[z_1] \oplus 1/2[z_2]$.
- *Lexicographic preferences*: Such a preference relation satisfies $I$ but not $C$ (an exercise!).
- *The worst case*: Such a preference relation does not satisfy $C$. In the two-prize case where $v(z_1) > v(z_2)$, $[z_1] \succ 1/2[z_1] \oplus 1/2[z_2]$. Viewed as points in $R^2_+$, we can rewrite this as $(1, 0) \succ (1/2, 1/2)$. Any neighborhood of $(1, 0)$ contains lotteries that are not strictly preferred to $(1/2, 1/2)$ and thus $C$ is not satisfied. Such a preference relation also does not satisfy $I$ ($[z_1] \succ [z_2]$ but $1/2[z_1] \oplus 1/2[z_2] \sim [z_2]$.)

## Utility Representation

By Debreu's theorem we know that for any relation $\succsim$ defined on the space of lotteries that satisfies $C$, there is a utility representation $U : L(Z) \to \Re$, continuous in the probabilities, such that $p \succsim q$ iff $U(p) \geq U(q)$. We will use the above axioms to isolate a family of preference relations and to derive a more *structured* utility function.

***Theorem (vNM):***

Let $\succsim$ be a preference relation over $L(Z)$ satisfying $I$ and $C$. There are numbers $(v(z))_{z \in Z}$ such that

$$p \succsim q \text{ iff } U(p) = \Sigma_{z \in Z} p(z)v(z) \geq U(q) = \Sigma_{z \in Z} q(z)v(z).$$

Note the distinction between $U(p)$ (the utility number of the lottery $p$) and $v(z)$ (called the Bernoulli numbers or the vNM utilities). The function $v$ is a utility function representing the preferences on $Z$ and is the building block for the construction of $U(p)$, a utility function representing the preferences on $L(Z)$. We will also often say that $v$ is a vNM utility function representing the preferences $\succsim$ over $L(Z)$.

For the proof of the theorem, we need the following lemma:

***Lemma:***

Let $\succsim$ be a preference over $L(Z)$ satisfying Axiom $I$. Let $x, y \in Z$ such that $[x] \succ [y]$ and $1 \geq \alpha > \beta \geq 0$. Then

$$\alpha x \oplus (1-\alpha)y \succ \beta x \oplus (1-\beta)y.$$

***Proof:***

If either $\alpha = 1$ or $\beta = 0$, the claim is implied by $I$. Otherwise, by $I$, $\alpha x \oplus (1-\alpha)y \succ [y]$. Using $I$ again we get: $\alpha x \oplus (1-\alpha)y \succ (\beta/\alpha)(\alpha x \oplus (1-\alpha)y) \oplus (1-\beta/\alpha)[y] = \beta x \oplus (1-\beta)y$.

***Proof of the theorem:***

Let $M$ and $m$ be the best and worst certain lotteries in $L(Z)$.

Consider first the case that $M \sim m$. It follows from $I^*$ that $p \sim m$ for any $p$ and thus $p \sim q$ for all $p,q \in L(Z)$. Choosing $v(z) = 0$ for all $z$ we have $\Sigma_{z \in Z} p(z)v(z) = 0$ for all $p \in L(Z)$. Thus, a constant utility function represents $\succsim$.

Now consider that $M \succ m$. By $C^*$ and the lemma, there is a single number $v(z) \in [0,1]$ such that $v(z)M \oplus (1-v(z))m \sim [z]$. (For example, $v(M) = 1$ and $v(m) = 0$). By $I^*$ we obtain that

$$p \sim (\Sigma_{z \in Z} p(z)v(z))M \oplus (1 - \Sigma_{z \in Z} p(z)v(z))m.$$

And by the lemma $p \succsim q$ iff $\Sigma_{z \in Z} p(z)v(z) \geq \Sigma_{z \in Z} q(z)v(z)$.

**The Uniqueness of vNM Utilities**

The vNM utilities are unique up to positive affine transformation (namely, multiplication by a positive number and adding any scalar) and are not invariant to arbitrary monotonic transformation. Consider a preference relation $\succsim$ defined over $L(Z)$ and define $v(z)$ as in the proof above. Of course, defining $w(z) = \alpha v(z) + \beta$ for all $z$ (for some $\alpha > 0$ and some $\beta$), the utility function $W(p) = \Sigma_{z \in Z} p(z)w(z)$ represents $\succsim$.

Furthermore, assume that $W(p) = \Sigma_z p(z)w(z)$ represents the preferences $\succsim$ as well. We will show that $w$ must be a positive affine transformation of $v$. To see this, let $\alpha > 0$ and $\beta$ satisfy

$$w(M) = \alpha v(M) + \beta \quad \text{and} \quad w(m) = \alpha v(m) + \beta$$

(the existence of $\alpha > 0$ and $\beta$ is guaranteed by $v(M) > v(m)$ and $w(M) > w(m)$). For any $z \in Z$ we have $[z] \sim v(z)M \oplus (1-v(z))m$, so it must be that

$$\begin{aligned} w(z) &= v(z)w(M) + (1-v(z))w(m) \\ &= v(z)[\alpha v(M) + \beta] + (1-v(z))[\alpha v(m) + \beta] \ = \alpha v(z) + \beta. \end{aligned}$$

## The Dutch Book Argument

There are those who consider expected utility maximization to be a normative principle. One of the arguments made to support this view is the following Dutch book argument. Assume that $L_1 \succ L_2$ but that $\alpha L \oplus (1-\alpha)L_2 \succ \alpha L \oplus (1-\alpha)L_1$. We can perform the following trick on the decision maker:

1. Take $\alpha L \oplus (1-\alpha)L_1$ (we can describe this as a contingency with random event $E$, which we both agree has probability $1-\alpha$).
2. Take instead $\alpha L \oplus (1-\alpha)L_2$, which you prefer (and you pay me something...).
3. Let us agree to replace $L_2$ with $L_1$ in case $E$ occurs (and you pay me something now).
4. Note that you hold $\alpha L \oplus (1-\alpha)L_1$.
5. Let us start from the beginning...

One possibility is to interpret the Dutch book as a "disequilibrium" in the decision maker's mind. Whatever he decides to do, he finds himself contradicting his own decision. If he has to choose between the two lotteries and makes up his mind to take $\alpha L \oplus (1-\alpha)L_1$, he recalls that he prefers $\alpha L \oplus (1-\alpha)L_2$, and if he is about to choose $\alpha L \oplus (1-\alpha)L_2$, he recalls that he will replace $L_2$ by $L_1$ if $E$ occurs.

• • •

### A Discussion of the Plausibility of the vNM Theory

Many experiments reveal systematic deviations from vNM assumptions. The most famous one is the *Allais paradox*. One version of it (see Kahneman and Tversky 2000) is presented here as follows.

Choose between

$$L_1 = 0.25[3,000] \oplus 0.75[0] \quad \text{and} \quad L_2 = 0.2[4,000] \oplus 0.8[0]$$

and now choose between

$$L_3 = 1[3,000] \quad \text{and} \quad L_4 = 0.8[4,000] \oplus 0.2[0].$$

Note that $L_1 = 0.25L_3 \oplus 0.75[0]$ and $L_2 = 0.25L_4 \oplus 0.75[0]$. Axiom $I$ requires that the preference between $L_1$ and $L_2$ be respectively the same as that between $L_3$ and $L_4$. However, a majority of people express the preferences $L_1 \prec L_2$ and an even larger majority express the preferences $L_3 \succ L_4$. Among about 140 graduate students at Princeton, Tel Aviv, and NYU (in 2002–4), although they were asked to respond to the above two choice problems on line one after the other, 67% chose $L_2$ while 80% chose $L_3$. This means that at least 47% of the students violated property $I$.

In my opinion, the last example demonstrates (again) the sensitivity of preference to the framing of the alternatives. When the lotteries $L_1$ and $L_2$ are presented as they are above, most prefer $L_2$. But, if we present $L_1$ and $L_2$ as the compound lotteries $L_1 = 0.25L_3 \oplus 0.75[0]$ and $L_2 = 0.25L_4 \oplus 0.75[0]$, most subjects prefer $L_1$ to $L_2$.

#### *Comment:*

In the proof of the vNM theorem we have seen that the independence axiom implies that if one is indifferent between $z$ and $z'$, one is also indifferent between $z$ and any lottery with $z$ and $z'$ as its prizes. This is not plausible in cases in which one takes into account the fairness of the random process that selects the prizes. For example, consider a parent in a situation where he has one gift and two children, $M$ and $Y$ (guess why I chose these letters). His options are to choose a lottery $L(p)$ that will award $M$ the gift with probability $p$ and $Y$ with probability $1 - p$. The parent does not favor one child over the other. The vNM approach "predicts" that he will be indifferent among all lotteries that determine who receives the gift, while common sense tells us he will strictly prefer $L(1/2)$.

## Bibliographic Notes

*Recommended readings*: Kreps 1990, 72–81 (115–122); Mas-Colell et al. 1995, chapter 6, A–B.

Expected utility theory is based on von Neumann and Morgenstern (1944). Kreps (1988) has an excellent presentation of the material. Machina (1987) remains a recommended survey of alternative theories. Kahneman and Tversky (1979) is a must reading for psychological criticism of expected utility theory. More recent material is covered in Kahneman and Tversky (2000).

# Problem Set 8

**Problem 1.** (*Standard*)
Consider the following preference relations that were described in the text: "The size of the support" and "Comparing the most likely prize."

a. Check carefully whether they satisfy axioms $I$ and $C$.
b. These preference relations are not immune to a certain "framing problem." Explain.

**Problem 2.** (*Standard.* Based on Markowitz 1959.)
One way to construct preferences over lotteries with monetary prizes is by evaluating each lottery $L$ on the basis of two numbers, $Ex(L)$, the expectation of $L$ and $var(L)$, $L$'s variance. Such a procedure may or may not be consistent with vNM assumptions.

a. Show that $u(L) = Ex(L) - (1/4)var(L)$ induces a preference relation that is not consistent with the vNM assumptions. (For example, consider the mixtures of each of the lotteries [1] and $0.5[0] \oplus 0.5[4]$ with the lottery $0.5[0] \oplus 0.5[2]$.)
b. Show that $u(L) = Ex(L) - (Ex(L))^2 - var(L)$ is consistent with vNM assumptions.

**Problem 3.** (*More difficult.* Based on Yaari 1987.)
In this problem you will encounter the functional of Quiggin and Yaari, one of the proposed alternatives to expected utility theory. Consider a world with the prizes $z_0, z_1, \ldots, z_K$. A decision maker attaches a number $v(z_k)$ to each $z_k$ such that $v(z_0) = 0 < v(z_1) < v(z_2) < \ldots < v(z_K)$ and evaluates each lottery $L$ by the number $U(L) = \Sigma_{k=1}^{K} f(G_L(z_k))[v(z_k) - v(z_{k-1})]$, where $f : [0, 1] \to [0, 1]$ is a continuous increasing function and $G_L(z_k) = \Sigma_{j \geq k} L(z_j)$. ($L(z)$ is the probability that the lottery $L$ yields $z$ and $G_L$ is the "anti-distribution" of $L$.)

a. Verify that for $f(x) = x$, $U(L)$ is the standard expected $v$-utility of $L$.
b. Show that the induced preference relation satisfies the continuity axiom but may not satisfy the independence axiom.
c. What are the difficulties with a functional form of the type $\Sigma_z f(p(z)) u(z)$?

**Problem 4.** (*Moderate*)
A decision maker has a preference relation $\succsim$ over the space of lotteries $L(Z)$ having a set of prizes $Z$. On Sunday he learns that on Monday he will be told whether he has to choose between $L_1$ and $L_2$ (probability $1 > \alpha > 0$) or between $L_3$ and $L_4$ (probability $1 - \alpha$). He will make his choice at that time.

Let us compare between two possible approaches the decision maker may take.

*Approach 1*: He delays his decision to Monday ("why bother with the decision now when I can make up my mind tomorrow. . . .").

*Approach 2*: He makes a contingent decision on Sunday regarding what he will do on Monday, that is, he instructs himself what to do if he faces the choice between $L_1$ and $L_2$ and what to do if he faces the choice between $L_3$ and $L_4$ ("On Monday morning I will be so busy. . . .").

a. Formulate approach 2 as a choice between lotteries.
b. Show that if the preferences of the decision maker satisfy the independence axiom, his choice under approach 2 will always be the same as under approach 1.

**Problem 5.** (*Difficult.* Bayesian updating.)
A decision maker has to choose an action from among a set $A$. The set of consequences is $Z$. For every action $a \in A$ the consequence $z^*$ is realized with probability $\alpha$, and any $z \in Z - \{z^*\}$ is realized with probability $(1 - \alpha)q(a, z)$.

a. Assume that once he has made his choice he is told that $z^*$ will not occur and is given a chance to change his decision. Show that if the decision maker obeys the Bayesian updating rule and follows vNM axioms, he will not change his decision.
b. Show that this is not necessarily the case if he does not obey the Bayesian rule or is using a nonexpected utility preference relation.

**Problem 6.** (*Standard*)
Assume there is a finite number of income levels and that the distribution over income levels is defined as the proportion of individuals at each level. In other words, we can think of a distribution as a lottery over income levels, with the probability of outcomes representing the proportions at each level. We often use the phrase "one distribution is more egalitarian than another."

a. Why is the von Neumann–Morgenstern independence axiom inappropriate for characterizing this type of relation?
b. Suggest a property that is appropriate, in your opinion, as an axiom for this type of relation. Give two examples of preference relations

that satisfy your property and express the desired relation in a logical fashion.

**Problem 7.** (*Difficult.* Based on Miyamoto, Wakker, Bleichrodt, and Peters 1998.)
A decision maker faces a trade-off between longevity and quality of life. His preference relation ranks lotteries on the set of all certain outcomes of the form $(q, t)$, defined as "a life of quality $q$ and length $t$" (where $q$ and $t$ are nonnegative numbers). Assume that the preference relation satisfies von Neumann–Morgenstern assumptions and that it also satisfies

- Indifference between "high" and "low" quality of life when longevity is 0.
- Expected longevity and quality of life are desirable.

a. Formalize the two assumptions.
b. Show that the preference relation derived from maximizing the expectation of the function $v(q)t$, where $v$ is a strictly increasing function and $v(q) > 0$ for all $q$, satisfies the assumptions.
c. Show that all preference relations satisfying the above assumptions can be represented by an expected utility function of the form $v(q)t$, where $v$ is a positive and increasing function.

**Problem 8.** (*Food for thought*)
Consider a decision maker who systematically calculates that $2 + 3 = 6$. Construct a "money pump" argument against him. Discuss the argument.

# Risk Aversion

## Lotteries with Monetary Prizes

We proceed to a discussion of a decision maker satisfying vNM assumptions where the space of prizes $Z$ is a set of real numbers and $a \in Z$ is interpreted as "receiving \$$a$." Note that in Lecture 8 we assumed the set $Z$ is finite; here, in contrast, we apply the expected utility approach to a set that is infinite. For simplicity we will still only consider lotteries with finite support. In other words, in this lecture, a lottery $p$ is a real function on $Z$ such that $p(z) \geq 0$ for all $z \in Z$, and there is a finite set $Y$ such that $\sum_{z \in Y} p(z) = 1$.

We will follow our general methodology and make special assumptions that fit the interpretation of the members of $Z$ as sums of money. Let $[x]$ be the lottery that yields the prize $x$ with certainty. We will say that $\succsim$ satisfies *monotonicity* if $a > b$ implies $[a] \succ [b]$. Thus, if $u$ is a vNM utility function representing a monotonic preference relation, then $u$ is a strictly increasing function.

An axiomatization (not presented here) of vNM preferences on an infinite space $Z$ requires strengthening of the continuity assumption so that if $p \succ q$, then small changes in the prizes, and not just in probabilities, leave the preferences unchanged. From here on we focus the discussion on preference relations over the space of lotteries for which there is a continuous function $u$ (referred to as a vNM utility function), such that the preference relation is represented by the function $Eu(p) = \sum_{z \in Z} p(z)u(z)$. This function assigns to the lottery $p$ the expectation of the random variable that receives the value $u(x)$ with a probability $p(x)$.

The following argument, called the *St. Petersburg Paradox*, is sometimes presented as a justification for assuming that vNM utility functions are bounded. Assume that a decision maker has an unbounded vNM utility function $u$. Consider playing the following "trick" on him:

1. Assume he possesses wealth $x_0$.
2. Offer him a lottery that will reduce his wealth to 0 with probability 1/2 and will increase his wealth to $x_1$ with probability 1/2 so that $u(x_0) < [u(0) + u(x_1)]/2$. By the unboundedness of $u$, there exists such an $x_1$.
3. If he loses, you are happy. If he is lucky, a moment before you give him $x_1$, offer him a lottery that will give him $x_2$ with probability 1/2 and 0 otherwise, where $x_2$ is such that $u(x_1) < [u(0) + u(x_2)]/2$.
4. And so on...

Our decision maker will find himself with wealth 0 with probability 1!

## First-Order Stochastic Domination

We say that *p first-order stochastically dominates q* (written as $pD_1q$) if $p \succsim q$ for any $\succsim$ on $L(Z)$ satisfying vNM assumptions as well as monotonicity in money. That is, $pD_1q$ if $Eu(p) \geq Eu(q)$ for all increasing $u$. Obviously, $pD_1q$ if the entire support of $p$ is to the right of the entire support of $q$. But, we are interested in a more interesting condition on a pair of lotteries $p$ and $q$, one that will be not only sufficient, but also necessary for $p$ to first-order stochastically dominate $q$.

For any lottery $p$ and a number $x$, define $G(p,x) = \sum_{z \geq x} p(z)$ (the probability that the lottery $p$ yields a prize at least as high as $x$). Denote by $F(p,x)$ the cumulative distribution function of $p$, that is, $F(p,x) = Probability\{z | z < x\}$.

***Claim:***

$pD_1q$ iff for all $x$, $G(p,x) \geq G(q,x)$ (alternatively, $pD_1q$ iff for all $x$, $F(p,x) \leq F(q,x)$). (See fig. 9.1.)

***Proof:***

Let $x_0 < x_1 < x_2 < \ldots < x_K$ be the prizes in the union of the supports of $p$ and $q$. First, note the following alternative expression for $Eu(p)$:

$$Eu(p) = \sum_{k \geq 0} p(x_k)u(x_k) = u(x_0) + \sum_{k \geq 1} G(p, x_k)(u(x_k) - u(x_{k-1})).$$

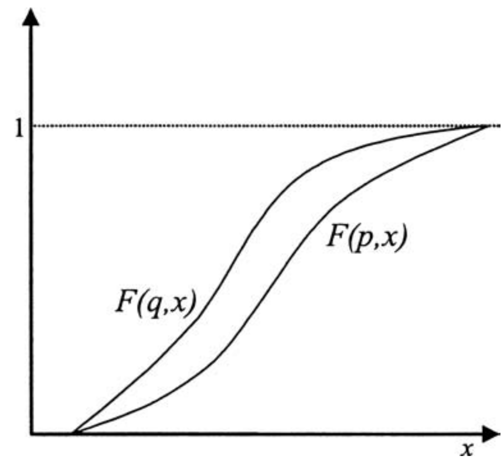



**Figure 9.1**
$p$ first-order stochastically dominates $q$.

Now, if $G(p, x_k) \geq G(q, x_k)$ for all $k$, then for all increasing $u$,

$$Eu(p) = u(x_0) + \sum_{k\geq 1} G(p, x_k)(u(x_k) - u(x_{k-1})) \geq$$

$$u(x_0) + \sum_{k\geq 1} G(q, x_k)(u(x_k) - u(x_{k-1})) = Eu(q).$$

Conversely, if there exists $k^*$ for which $G(p, x_{k^*}) < G(q, x_{k^*})$, then we can find an increasing function $u$ so that $Eu(p) < Eu(q)$, by setting $u(x_{k^*}) - u(x_{k^*-1})$ to be very large and the other increments to be very small.

We have just discussed the simplest example of questions of the type: "Given a set of preference relations on $L(Z)$, for what pairs $p, q \in L(Z)$ is $p \succsim q$ for all $\succsim$ in the set?" In the problem set you will discuss another example of this kind of question.

## Risk Aversion

We say that $\succsim$ is *risk averse* if for any lottery $p$, $[Ep] \succsim p$.

We will see now that for a decision maker with preferences $\succsim$ obeying the vNM axioms, risk aversion is closely related to the concavity of the vNM utility function representing $\succsim$.

First recall some basic properties of concave functions (if you are not familiar with those properties, this will be an excellent opportunity for you to prove them yourself):

1. An increasing and concave function must be continuous (but not necessarily differentiable).
2. The *Jensen Inequality*: If $u$ is concave, then for any finite sequence $(\alpha_k)_{k=1,\dots,K}$ of positive numbers that sum up to 1, $u(\sum_{k=1}^{K} \alpha_k x_k) \geq \sum_{k=1}^{K} \alpha_k u(x_k)$.
3. The *Three Strings Lemma*: For any $a < b < c$ we have

$$[u(c) - u(b)]/(c - b) \leq [u(c) - u(a)]/(c - a) \leq [u(b) - u(a)]/(b - a).$$

4. If $u$ is differentiable, then for any $a < c$, $u'(a) \geq u'(c)$, and thus $u''(x) \leq 0$ for all $x$.

***Claim:***

Let $\succsim$ be a preference on $L(Z)$ represented by the vNM utility function $u$. The preference relation $\succsim$ is risk averse iff $u$ is concave.

***Proof:***

Assume that $u$ is concave. By the Jensen Inequality, for any lottery $p$, $u(E(p)) \geq Eu(p)$ and thus $[E(p)] \succsim p$.

Assume that $\succsim$ is risk averse and that $u$ represents $\succsim$. For all $\alpha \in (0, 1)$ and for all $x, y \in Z$, we have by risk aversion $[\alpha x + (1 - \alpha)y] \succsim \alpha x \oplus (1 - \alpha)y$ and thus $u(\alpha x + (1 - \alpha)y) \geq \alpha u(x) + (1 - \alpha)u(y)$, that is, $u$ is concave.

### Certainty Equivalence and the Risk Premium

Let $E(p)$ be the expectation of the lottery $p$, that is, $E(p) = \sum_{z \in Z} p(z)z$. Given a preference relation $\succsim$ over the space $L(Z)$, the *certainty equivalence* of a lottery $p$, $CE(p)$, is a prize satisfying $[CE(p)] \sim p$. (To justify the existence of $CE(p)$ we need to assume that $\succsim$ is monotonic and continuous in the sense that if $p \succ q$, the inequality is maintained if we change both lotteries' probabilities *and* prizes a "little bit"). The

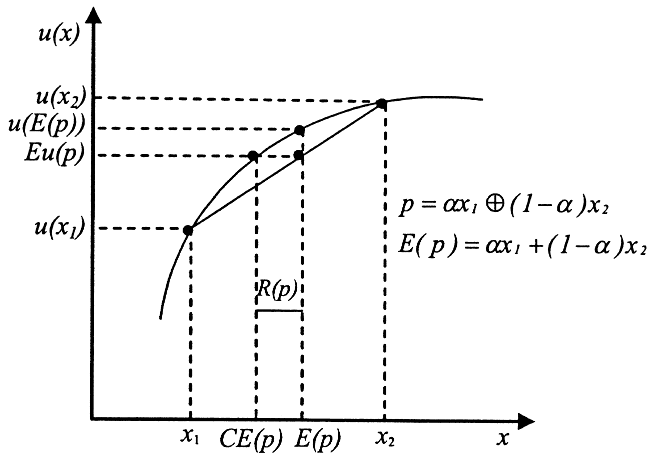



**Figure 9.2**
CE and risk premium.

*risk premium* of $p$ is the difference $R(P) = E(p) - CE(p)$. By definition, the preferences are risk averse if and only if $R(p) \geq 0$ for all $p$. (See fig. 9.2.)

### The "More Risk Averse" Relation

We wish to formalize the statement "one decision maker *is more risk averse* than another." To understand the logic of the following definitions let us start with an analogous phrase: What is the meaning of the statement "$A$ is more war averse than $B$"? One possible meaning is that whenever $A$ is ready to go to war, $B$ is as well. Another possible meaning is that when facing the threat of war, $A$ is ready to agree to a less attractive compromise in order to prevent war than $B$. The following two definitions are analogous to these two interpretations.

1. The preference relation $\succsim_1$ *is more risk averse than* $\succsim_2$ if for any lottery $p$ and degenerate lottery $c$, $p \succsim_1 c$ implies that $p \succsim_2 c$.

In case the preferences are monotonic, we have a second definition:

2. The preference relation $\succsim_1$ *is more risk averse than* $\succsim_2$ if $CE_1(p) \leq CE_2(p)$ for all $p$.

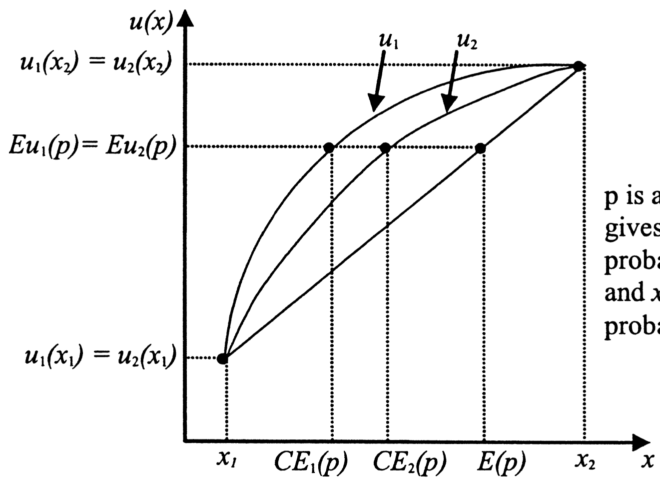



**Figure 9.3**
1 is more risk averse than 2.

In case the preferences satisfy vNM assumptions, we have a third definition:

3. Let $u_1$ and $u_2$ be vNM utility functions representing $\succsim_1$ and $\succsim_2$, respectively. The preference relation $\succsim_1$ is more risk averse than $\succsim_2$ if the function $\varphi$, defined by $u_1(t) = \varphi(u_2(t))$, is concave.

I find definition (1) particularly attractive since it is meaningful in any space of prizes (not only those in which consequences are numerical) and for a general set of preferences (and not only those satisfying vNM assumptions). (See fig. 9.3.)

***Claim:***

If both $\succsim_1$ and $\succsim_2$ are preference relations on $L(Z)$ represented by increasing and continuous vNM utility functions, then the three definitions are equivalent.

***Proof:***

- If (2), then (1).
  Assume (2). Then, if $p \succsim_1 [c]$, it has to be that $[CE_1(p)] \succsim_1 [c]$ and thus $CE_1(p) \geq c$, which implies also that $CE_2(p) \geq c$, that is, $p \succsim_2 [c]$.

- If (3) then (2).
  By definition, $Eu_i(p) = u_i(CE_i(p))$. Thus, $CE_i(p) = u_i^{-1}(Eu_i(p))$. If $\varphi = u_1 u_2^{-1}$ is concave, then by the Jensen Inequality:
  $$u_1(CE_2(p)) = u_1(u_2^{-1}(Eu_2(p)) = \varphi(\sum_k p(x_k)u_2(x_k)) \geq$$
  $$(\sum_k p(x_k)\varphi u_2(x_k)) = \sum_k p(x_k)u_1(x_k) = E(u_1(p)) = u_1(CE_1(p)).$$
  Thus, $CE_2(p) \geq CE_1(p)$.
- If (1), then (3).
  Consider three numbers $u_2(x) < u_2(y) < u_2(z)$ in the range of $u_2$ and let $\lambda \in (0, 1)$ satisfy $u_2(y) = \lambda u_2(x) + (1 - \lambda)u_2(z)$. Let us see that $u_1(y) \geq \lambda u_1(x) + (1 - \lambda)u_1(z)$.
  If $u_1(y) < \lambda u_1(x) + (1 - \lambda)u_1(z)$, then for some $w > y$ close enough to $y$, we have both $w \prec_1 \lambda x \oplus (1 - \lambda)z$ and $w \succ_2 \lambda x \oplus (1 - \lambda)z$, which contradicts (1). Thus, $y \succsim_1 \lambda x \oplus (1 - \lambda)z$ and $u_1(y) \geq \lambda u_1(x) + (1 - \lambda)u_1(z)$, from which it follows that $\varphi(u_2(y)) \geq \lambda\varphi(u_2(x)) + (1 - \lambda)\varphi(u_2(z))$. Thus, $\varphi$ is concave.

## The Coefficient of Absolute Risk Aversion

The following is another definition of the relation "more risk averse" applied to the case in which vNM utility functions are differentiable:

4. Let $u_1$ and $u_2$ be differentiable vNM utility functions representing $\succsim_1$ and $\succsim_2$, respectively. The preference relation $\succsim_1$ *is more risk averse than* $\succsim_2$ if $r_2(x) \leq r_1(x)$ for all $x$, where $r_i(x) = -u_i''(x)/u_i'(x)$.

The number $r(x) = -u''(x)/u'(x)$ is called the *coefficient of absolute risk aversion* of $u$ at $x$. We will see that a higher coefficient of absolute risk aversion means a more risk-averse decision maker.

To see that (3) and (4) are equivalent, note the following chain of equivalences:

- Definition (3) (that is, $u_1 u_2^{-1}$ is concave) is satisfied iff
- the function $d/dt[u_1(u_2^{-1}(t))]$ is nonincreasing in $t$ iff
- $u_1'(u_2^{-1}(t))/u_2'(u_2^{-1}(t))$ is nonincreasing in $t$ iff (since $(\varphi^{-1})'(t) = 1/\varphi'(\varphi^{-1}(t))$)
- $u_1'(x)/u_2'(x)$ is nonincreasing in $x$ (since $u_2^{-1}(t)$ is increasing in $t$) iff

- $log[(u'_1/u'_2)(x)] = logu'_1(x) - logu'_2(x)$ is nonincreasing in $x$ iff
- the derivative of $logu'_1(x) - logu'_2(x)$ is nonpositive iff
- $r_2(x) - r_1(x) \le 0$ for all $x$ where $r_i(x) = -u''_i(x)/u'_i(x)$ iff
- definition (4) is satisfied.

For a better understating of the coefficient of absolute risk aversion, it is useful to look at the preferences on the restricted domain of lotteries of the type $(x_1, x_2) = px_1 \oplus (1-p)x_2$, where the probability $p$ is fixed. Denote by $u$ a differentiable vNM utility function that represents a risk-averse preference.

Let $x_2 = \psi(x_1)$ be the function describing the indifference curve through $(t, t)$, the point representing $[t]$. It follows from risk aversion that all lotteries with expectation $t$, that is, all lotteries on the line $\{(x_1, x_2) | \; px_1 + (1-p)x_2 = t\}$, are not above the indifference curve through $(t, t)$. Thus, $\psi'(x_1) = -p/(1-p)$.

By definition of $u$ as a vNM utility function representing the preferences over the space of lotteries, we have $pu(x_1) + (1-p)u(\psi(x_1)) = u(t)$. Taking the derivative with respect to $x_1$, we obtain $pu'(x_1) + (1-p)u'(\psi(x_1))\psi'(x_1) = 0$. Taking the derivative with respect to $x_1$ once again, we obtain

$$pu''(x_1) + (1-p)u''(\psi(x_1))[\psi'(x_1)]^2 + (1-p)u'(\psi(x_1))\psi''(x_1) = 0.$$

At $x_1 = t$ we have

$$pu''(t) + u''(t)p^2/(1-p) + (1-p)u'(t)\psi''(t) = 0.$$

Therefore,

$$\psi''(t) = -u''(t)/u'(t)[p/(1-p)^2] = r(t)[p/(1-p)^2].$$

That is, the second derivative of the indifference curve through the certain lottery $t$ is $r(t)[p/(1-p)^2]$.

Note that on this restricted space of lotteries, $\succsim_1$ is more risk averse than $\succsim_2$ in the sense of definition (1) iff the indifference curve of $\succsim_1$ through $(t, t)$, denoted by $\psi_1$, is never below the indifference curve of $\succsim_2$ through $(t, t)$, denoted by $\psi_2$. Combined with $\psi'_1(t) = \psi'_2(t)$, we obtain that $\psi''_1(t) \ge \psi''_2(t)$ and thus $r_2(t) \le r_1(t)$. (See fig. 9.4.)

## The Doctrine of Consequentialism

Conduct the following "thought experiment":

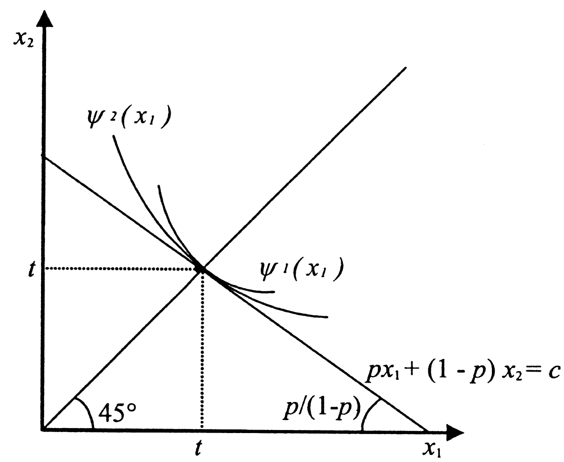



**Figure 9.4**
1 is more risk averse than 2.

You have \$2000 in your bank account. You have to choose between

1. a sure loss of \$500
   and
2. a lottery in which you lose \$1000 with probability 1/2 and lose 0 with probability 1/2.

What is your choice?

Now assume that you have \$1000 in your account and that you have to choose between

3. a certain gain of \$500
   and
4. a lottery in which you win \$1000 with probability 1/2 and win 0 with probability 1/2.

What is your choice?

In the first case, most people preferred the lottery to the certain prize (chose (2)), while in the second case most people preferred the sure prize (chose (3)). Such a preference does not conflict with expected utility theory if we interpret a prize to reflect a "monetary change." However, if we assume that the decision maker takes the final wealth levels to be his prizes, we have a problem: in terms of final wealth levels, both choices can be presented as being between a sure prize of \$1500 and a lottery that yields \$2000 or \$1000 with probability 1/2 each.

Nevertheless, in the economic literature it is usually assumed that a decision maker's preferences over wealth changes are induced from his preferences with regard to "final wealth levels." Formally, when starting with wealth $w$, denote by $\succsim_w$ the decision maker's preferences over lotteries in which the prizes are interpreted as "changes" in wealth. By the *doctrine of consequentialism* all relations $\succsim_w$ are derived from the same preference relation, $\succsim$, defined over the "final wealth levels" by $p \succsim_w q$ iff $w + p \succsim w + q$ (where $w + p$ is the lottery that awards a prize $w + x$ with probability $p(x)$). If $\succsim$ is represented by a vNM utility function $u$, this doctrine implies that for all $w$, the function $v_w(x) = u(w + x)$ is a vNM utility function representing the preferences $\succsim_w$.

## Invariance to Wealth

We say that the preference relation $\succsim$ exhibits *invariance to wealth* (in the literature it is often called *constant absolute risk aversion*) if the induced preference relation $\succsim_w$ is independent of $w$, that is, $(w + L_1) \succsim (w + L_2)$ is true or false independent of $w$.

We will see that if $u$ is a continuous vNM utility function representing preferences $\succsim$, which exhibit risk aversion and invariance to wealth, then $u$ must be exponential.

Let us first confine ourselves to the $\Delta - grid$ prize space, $Z = \{x \mid x = n\Delta$ for some integer $n\}$. This domain has a special meaning when we take $\Delta$ to be the smallest (indivisible) unit of money.

By continuity of $u$, for any wealth level $x$ there is a number $q$ such that $(1 - q)(x - \Delta) \oplus q(x + \Delta) \sim x$. By the invariance to wealth, $q$ is independent of $x$. Thus, we have $u(x + \Delta) - u(x) = ((1 - q)/q)[u(x) - u(x - \Delta)]$ for all $x$. This means that the increments in the function $u$, when $x$ is increased by $\Delta$, constitute a geometric sequence with a factor of $(1 - q)/q$ (where $q$ might depend on $\Delta$). We conclude that the function $u$, defined on the $\Delta - grid$, must be an affine transformation of $((1 - q)/q)^{x/\Delta}$.

Let us now return to the case of $Z = \Re$ and look at the preferences over the restricted space of all lotteries of the type $(x_1, x_2) = px_1 \oplus (1 - p)x_2$ for some arbitrary fixed probability number $p \in (0, 1)$. A necessary condition for $\succsim$ to exhibit constant absolute risk aversion is that the indifference curve through $(t, t)$ is the same as that through $(0, 0)$, shifted in the direction of $(t, t)$. In other words, denoting the indifference through $(t, t)$ by $x_2 = \psi_t(x_1)$, we have $\psi_t(x_1) = \psi_0(x_1 - t) + t$.

Assuming that the function $u$ is differentiable, we derive that $\psi_t''(t) = \psi_0''(0)$. We have already seen that $\psi_t''(t) = -[p/(1-p)^2]$ $[u_i''(t)/u_i'(t)]$ and thus there exists a constant $\alpha$ such that $-u''(t)/u'(t) = \alpha$ for all $t$. This implies that $[log u'(t)]' = -\alpha$ for all $t$ and $log u'(t) = -\alpha t + \beta$ for some $\beta$. It follows that $u'(t) = e^{-\alpha t + \beta}$. If $\alpha = 0$, the function $u(t)$ must be linear (implying risk neutrality). If $\alpha \neq 0$, it must be that $u(t) = ce^{-\alpha t} + d$ for some $c$ and $d$.

To conclude, if $u$ is a vNM continuous utility function representing preferences that are monotonic and exhibit both risk aversion and invariance to wealth, then $u$ is an affine transformation of either the function $t$ or a function $-e^{-\alpha t}$ (with $\alpha > 0$).

## Critique of the Doctrine of Consequentialism

Denote by $1/2(-D) \oplus 1/2(+G)$ the lottery in which there is an equal probability of gaining \$$G$ and losing \$$D$. Consider a risk-averse decision maker who likes money, obeys expected utility theory, and adheres to the doctrine of consequentialism. Matthew Rabin noted that if such a decision maker turns down the lottery $L = 1/2(-10) \oplus 1/2(+11)$, at any wealth level between \$0 and \$5000 (a quite plausible assumption), then at the wealth level \$4000 he must reject the lottery $1/2(-100) \oplus 1/2(+71000)$ (a quite ridiculous conclusion).

The intuition for this observation is quite simple. Let $\Delta$ be the marginal utility of one dollar at the wealth level $w$. If $L$ is rejected at $w$, then it must be that the marginal utility level at $w + 21$ is not more than $(21/22)\Delta$. To see this, note that the marginal utility at $w + 21$ is (by the concavity of $u$) not greater than $[u(w + 21) - u(w + 10)]/11$. Since $L$ is rejected in $w$, $u(w + 10) \geq [u(w) + u(w + 21)]/2$ and thus the marginal utility at $w + 21$ is not greater than

$$\{u(w + 21) - [u(w + 21) + u(w + 0)]/2\}/11$$
$$= [u(w + 21) - u(w + 0)]/22 \leq (21/22)\Delta.$$

Thus, the sequence of marginal utilities within the domain of wealth levels in which $L$ is rejected falls at a geometric rate. This implies that for the lottery $1/2(-D) \oplus 1/2(+G)$ to be accepted even for a relatively low $D$, one would need a huge $G$.

What conclusions should we draw from this observation? In my opinion, in contrast to what some scholars claim, this is not a refutation of expected utility theory. Rabin's argument relies on the doctrine of consequentialism, which is not a part of expected utility theory. Expected utility theory is invariant to the interpretation of the prizes. Independently of the theory of decision making under uncertainty that we use, the set of prizes should be the set of consequences in the mind of the decision maker. Thus, it is equally reasonable to assume the consequences are "wealth changes" or "final wealth levels."

I treat Rabin's argument as further evidence of the empirically problematic nature of the doctrine of consequentialism according to which the decision maker makes *all* decisions having in mind a preference relation over *the same* set of final consequences. It also demonstrates how carefully we should tread when trying to estimate real life agents' utility functions. The practice of estimating an economic agent's risk aversion parameters for small lotteries might lead to misleading conclusions if such estimates are used to characterize the decision maker's preferences regarding lotteries over large sums.

## Bibliographic Notes

*Recommended readings*: Kreps 1990, 81–98; Mas-Colell et al. 1995, Chapter 6, C–D.

The measures of risk aversion are taken from Arrow (1970) and Pratt (1964). For the psychological literature discussed here, see Kahneman and Tversky (1979) and Kahneman and Tversky (2000).

The St. Petersburg Paradox was suggested by Daniel Bernoulli in 1738 (see Bernoulli 1954). The notion of stochastic domination was introduced into the economic literature by Rothschild and Stiglitz (1970). Rabin's argument is based on Rabin (2000).

# Problem Set 9

**Problem 1.** (*Standard.* Based on Rothschild and Stiglitz 1970.)
We say that *p second-order stochastically dominates q* and denote it by $pD_2q$ if $p \succsim q$ for all preferences $\succsim$ satisfying the vNM assumptions, monotonicity *and* risk aversion.

a. Explain why $pD_1q$ implies $pD_2q$.
b. Let $p$ and $\varepsilon$ be lotteries. Define $p+\varepsilon$ to be the lottery that yields the prize $t$ with the probability $\Sigma_{\alpha+\beta=t}p(\alpha)\varepsilon(\beta)$. Interpret $p+\varepsilon$. Show that if $\varepsilon$ is a lottery with expectation 0, then for all $p$, $pD_2(p+\varepsilon)$.
c. (More difficult) Show that $pD_2q$ if and only if for all $t<K$, $\Sigma_{k=0}^{t}$ $[G(p,x_{k+1})-G(q,x_{k+1})][x_{k+1}-x_k] \geq 0$ where $x_0 < \ldots < x_K$ are all the prizes in the support of either $p$ or $q$ and $G(p,x)=\Sigma_{z\geq x}p(z)$.

**Problem 2.** (*Standard.* Based on Slovic and Lichtenstein 1968.)
Consider a phenomenon called *preference reversal*. Let $L_1 = 8/9[\$4] \oplus 1/9[\$0]$ and $L_2 = 1/9[\$40] \oplus 8/9[\$0]$.

a. What is the maximal amount you are willing to pay for $L_1$? For $L_2$?
b. What lottery do you prefer?
c. Discuss the "typical" answer that ranks $L_1$ as superior to $L_2$ but attaches a lower value to $L_1$ (see Slovic, Tversky and Kahneman 1990).

**Problem 3.** (*Standard*)
Consider a consumer's preference over $K$-tuples of $K$ uncertain assets. Denote the random return on the $k$th asset by $Z_k$. Assume that the random variables $(Z_1,\ldots,Z_K)$ are independent and take positive values with probability 1. If the consumer buys the combination of assets $(x_1,\ldots,x_K)$ and if the vector of realized returns is $(z_1,\ldots,z_K)$, then the consumer's total wealth is $\sum_{k=1}^{K} z_k x_k$. Assume that the consumer satisfies vNM assumptions, that is, there is a function $v$ (over the sum of his returns) so that he maximizes the expected value of $v$. Assume that $v$ is increasing and concave. The consumer preferences over the space of the lotteries induce preferences on the space of investments. Show that the induced preferences are monotonic and convex.

**Problem 4.** (*Standard.* Based on Rubinstein 2002.)
Adam lives in the Garden of Eden and eats only apples. Time in the garden is discrete ($t = 1, 2, \ldots$) and apples are eaten only in discrete units. Adam possesses preferences over the set of streams of apple consumption. Assume that Adam:

a. Likes to eat up to 2 apples a day and cannot bear to eat 3 apples a day.
b. Is impatient. He will be delighted to increase his consumption at day $t$ from 0 to 1 or from 1 to 2 apples at the expense of an apple he is promised a day later.
c. At any period in which he does not have an apple, he prefers to get one apple immediately in exchange for two apples tomorrow.
d. Cares only about his consumption in the first 120 years of his life.

Show that if (poor) Adam is offered a stream of 2 apples starting in period 19 for the rest of his life (assuming he does not expect to live more than 120 years), he would be willing to exchange that offer for one apple given right away.

LECTURE 10

# Social Choice

## Aggregation of Orderings

When a rational decision maker forms a preference relation, it is often on the basis of more primitive relations. For example, the choice of a PC may depend on considerations such as "size of memory," "ranking by PC magazine," and "price." Each of these considerations expresses a preference relation on the set of PCs. In this lecture we look at some of the logical properties and problems that arise in the formation of preferences on the basis of more primitive preference relations.

Although the aggregation of preference relations can be thought of in a context of a single individual's decision making, the classic context in which preference aggregation is discussed is "social choice," where the "will of the people" is thought of as an aggregation of the preference relations held by members of society.

The foundations of social choice theory lie in the "Paradox of Voting." Let $X = \{a, b, c\}$ be a set of alternatives. Consider a society that consists of three members called 1, 2, and 3. Their rankings of $X$ are $a \succ_1 b \succ_1 c$; $b \succ_2 c \succ_2 a$, and $c \succ_3 a \succ_3 b$. A natural criterion for the determination of collective opinion on the basis of individuals' preference relations is the *majority rule*. According to the majority rule, $a \succ b$, $b \succ c$, and $c \succ a$, which conflicts with the transitivity of the social ordering. Note that although the majority rule does not induce a transitive social relation for *all* profiles of individuals' preference relations, transitivity is guaranteed if we restrict ourselves to a smaller domain of profiles (see problem 3 in the problem set).

The interest in social choice in economics is motivated by the recognition that explicit methods for the aggregation of preference relations are essential for doing any *welfare economics*. The theory is also related to the design of *voting systems*, which are methods for determining social action on the basis of individuals' preferences.

### The Basic Model

A basic model of social choice consists of the following:

- $X$: a set of social *alternatives*.
- $N$: a finite set of *individuals* (denote the number of elements in $N$ by $n$).
- $\succ_i$: individual $i$'s linear ordering on $X$ (a linear ordering is a preference relation with no indifferences, i.e., for no $x \neq y$, $x \sim_i y$).
- *Profile*: An $n$-tuple of orderings $(\succ_1, \ldots, \succ_n)$ interpreted as a certain "state of society."
- *SWF* (*Social Welfare Function*): A function that assigns a single (social) preference (*not* necessarily a linear ordering) to every profile.

Note that

1. The assumption that the domain of an SWF includes only strict preferences is made only for simplicity of presentation.
2. An SWF attaches a preference relation to *every* possible profile and not just to a single profile.
3. The SWF aggregation of preference relations is required to produce a complete preference relation. An alternative concept, called Social Choice Function, attaches a social alternative, interpreted as the society's choice, to every profile of preference relations.
4. An SWF aggregates only ordinal preference relations. The framework does not allow us to make a statement such as "the society prefers $a$ to $b$ since agent 1 prefers $b$ to $a$ but agent 2 prefers $a$ to $b$ much more."
5. In this model we cannot express a consideration of the type "I prefer what society prefers."
6. The elements in $X$ are social alternatives. Thus, an individual's preferences may exhibit considerations of fairness and concern about other individuals' well-being.

### Examples

Let us consider some examples of aggregation procedures. We will often use $\succsim$ as a short form of $F(\succ_1, \ldots, \succ_n)$.

1. $F(\succ_1, \ldots, \succ_n) = \succsim^*$ for some ordering $\succsim^*$. (This is a degenerate SWF that does not account for the individuals' preferences.)
2. Define $x \to z$ if a majority of individuals prefer $x$ to $z$. Order the alternatives by the number of "victories" they score, that is, $x \succsim y$ if $|\{z|x \to z\}| \geq |\{z|y \to z\}|$.
3. For $X = \{a, b\}$, $a \succsim b$ unless 2/3 of the individuals prefer $b$ to $a$.
4. "*The anti-dictator*": There is an individual $i$ so that $x$ is preferred to $y$ if and only if $y \succ_i x$.
5. Define $d(\succ; \succ_1, \ldots, \succ_n)$ as the number of $(x, y, i)$ for which $x \succ_i y$ and $y \succ x$. The function $d$ can be interpreted as the sum of the distances between the preference relation $\succ$ and the $n$ preference relations of the individuals. Choose $F(\succ_1, \ldots, \succ_n)$ to be an ordering that minimizes $d(\succ; \succ_1, \ldots, \succ_n)$ (ties are broken arbitrarily).
6. Let $F(\succ_1, \ldots, \succ_n)$ be the ordering that is the most common among $(\succ_1, \ldots, \succ_n)$ (with ties broken in some predetermined way).

## Axioms

Once again we use the axiomatization methodology. We suggest a set of axioms on social welfare functions and study their implications. Let $F$ be an SWF.

### *Condition Par (Pareto):*

For all $x, y \in X$ and for every profile $(\succ_i)_{i \in N}$, if $x \succ_i y$ for all $i$ then $x \succ y$.

The Pareto axiom requires that if all individuals prefer one alternative over the other, then the social preferences agree with the individuals'.

### *Condition IIA (Independence of Irrelevant Alternatives):*

For any pair $x, y \in X$ and any two profiles $(\succ_i)_{i \in N}$ and $(\succ'_i)_{i \in N}$ if for all i, $x \succ_i y$ iff $x \succ'_i y$, then $x \succsim y$ iff $x \succsim' y$.

The IIA condition requires that if two profiles agree on the relative rankings of two particular alternatives, then the social preferences attached to the two profiles also agree in their relative ranking of the two alternatives.

Notice that IIA allows an SWF to apply one criterion when comparing $a$ to $b$ and another when comparing $c$ to $d$. For example, the simple social preference between $a$ and $b$ can be determined according to majority rule while that between $c$ and $d$ requires a 2/3 majority.

Condition IIA is sufficient for Arrow's theorem. However, for the sake of simplifying the proof in this presentation, we will make do with a stronger requirement:

***Condition I* (Independence of Irrelevant Alternatives + Neutrality):***

For all $a, b, c, d \in X$, and for any profiles $\succ$ and $\succ'$ if for all $i$, $a \succ_i b$ iff $c \succ'_i d$, then $a \succsim b$ iff $c \succsim' d$.

In other words, in addition to what is required by *IIA*, condition $I^*$ requires that the criterion that determines the social preference between $a$ and $b$ be applied to *any* pair of alternatives.

### Arrow's Impossibility Theorem

***Theorem (Arrow):***

If $|X| \geq 3$, then any SWF $F$ that satisfies conditions *Par* and $I^*$ is dictatorial, that is, there is some $i^*$ such that $F(\succ_1, \ldots, \succ_n) \equiv \succ_{i^*}$.

We can break the theorem's assumptions into four: *Par*, $I^*$, *Transitivity* (of the social ordering), and $|X| \geq 3$. Before we move on to the proof, let us show that the assumptions are *independent*. Namely, for each of the four assumptions, we give an example of a nondictatorial SWF, demonstrating the theorem would not hold if that assumption were omitted.

- *Par*: An anti-dictator SWF satisfies $I^*$ but not *Par*.
- $I^*$: Consider the Borda Rule. Let $w(1) > w(2) > \ldots > w(|X|)$ be a fixed profile of weights. Say that $i$ assigns to $x$ the score

$w(k)$ if $x$ appears in the $k$'th place in $\succ_i$. Attach to $x$ the sum of the weights assigned to $x$ by the $n$ individuals and rank the alternatives by those sums. The Borda rule is an SWF satisfying *Par* but not $I^*$.

- *Transitivity of the Social Order:* The majority rule satisfies all assumptions but can induce a relation which is not transitive.
- $|X| \geq 3$: For $|X| = 2$ the majority rule satisfies *Par* and $I^*$ and induces (a trivial) transitive relation.

## Proof of Arrow's Impossibility Theorem

Let $F$ be an SWF that satisfies *Par* and $I^*$. Hereinafter, we denote the relation $F(\succ_1, \ldots, \succ_n)$ by $\succsim$.

Given the SWF we say that

- a coalition $G$ is *decisive* if for all $x$,$y$, [for all $i \in G$, $x \succ_i y$] implies [$x \succ y$], and
- a coalition $G$ is *almost decisive* if for all $x$,$y$, [for all $i \in G$, $x \succ_i y$ and for all $j \notin G$ $y \succ_j x$] implies [$x \succ y$].

Note that if $G$ is decisive it is almost decisive since the "almost decisiveness" refers only to the subset of profiles where all members of $G$ prefer $x$ to $y$ and all members of $N - G$ prefer $y$ to $x$.

### *Field Expansion Lemma:*

If $G$ is almost decisive, then $G$ is decisive.

### *Proof:*

We have to show that for any $x$,$y$ and for any profile $(\succ'_i)_{i \in N}$ for which $x \succ'_i y$ for all $i \in G$, the preference $F(\succ'_1, \ldots, \succ'_n)$ determines $x$ to be superior to $y$. By $I^*$ it is sufficient to show that for one pair of social alternatives $a$ and $b$, and for one profile $(\succ_i)_{i \in N}$ that agrees with the profile $(\succ'_i)_{i \in N}$ on the pair $\{a, b\}$, the preference $F(\succ_1, \ldots, \succ_n)$ determines $a$ to be preferred to $b$.

Let $c$ be a third alternative. Let $(\succ_i)_{i \in N}$ be a profile satisfying $a \succ_i b$ iff $a \succ'_i b$ for all $i$, and for all $i \in G$, $a \succ_i c \succ_i b \succ_i x$ for every $x \in X - \{a, b, c\}$ and for all $j \notin G$, $c \succ_j y \succ_j x$ for every $y \in \{a, b\}$ and for every $x \in X - \{a, b, c\}$.

Since $G$ is almost decisive, $a \succ c$. By *Par*, $c \succ b$, therefore, $a \succ b$ by transitivity.

***Group Contraction Lemma:***

If $G$ is decisive and $|G| \geq 2$, then there exists $G' \subset G$ such that $G'$ is decisive.

***Proof:***

Let $G = G_1 \cup G_2$, where $G_1$ and $G_2$ are nonempty and $G_1 \cap G_2 = \emptyset$. By the Field Expansion Lemma it is enough to show that $G_1$ or $G_2$ is almost decisive.

Take three alternatives $a, b$, and $c$ and a profile of preference relations $(\succ_i)_{i \in N}$ satisfying

- for all $i \in G_1$, $c \succ_i a \succ_i b$, and
- for all $i \in G_2$, $a \succ_i b \succ_i c$, and
- for all other $i$, $b \succ_i c \succ_i a$.

If $G_1$ is not almost decisive, then there are $x$ and $y$ and a profile $(\succ'_i)_{i \in N}$ such that $x \succ'_i y$ for all $i \in G_1$ and $y \succ'_i x$ for all $i \notin G_1$, such that $F(\succ'_1, \ldots, \succ'_n)$ determines $y$ to be at least as preferable as $x$. Therefore, by $I^*$, $b \succsim c$.

Similarly, if $G_2$ is not almost decisive, then $c \succsim a$. Thus, by transitivity $b \succsim a$, but since $G$ is decisive, $a \succ b$, a contradiction. Thus, $G_1$ or $G_2$ is almost decisive.

***Proof of the Theorem:***

By *Par*, the set $N$ is decisive. By the Group Contraction Lemma, every decisive set that includes more than one member has a proper subset that is decisive. Thus, there is a set $\{i^*\}$ that is decisive, namely, $F(\succ_1, \ldots, \succ_n) \equiv \succ_{i^*}$.

## Related Issues

Arrow's theorem was the starting point for a huge literature. We mention three other impossibility results.

1. *Monotonicity* is another axiom that has been widely discussed in the literature. Consider a "change" in a profile so that an alternative $a$, which individual $i$ ranked below $b$, is now ranked by $i$ above $b$. Monotonicity requires that there is no alternative $c$ such that this change deteriorates the ranking of $a$ vs. $c$. Muller and Satterthwaite (1977)'s theorem shows that the only SWF's satisfying *Par* and monotonicity are dictatorships.

2. An SWF specifies a preference relation for every profile. A *social choice function* attaches an alternative to every profile. The most striking theorem proved in this framework is the Gibbard-Satterthwaite theorem. It states that any social choice function $C$ satisfying the condition that it is never worthwhile for an individual to mispresent his preferences, namely, it is never that $C(\succ_1, \ldots, \succ_i' , \ldots, \succ_n) \succ_i C(\succ_1, \ldots, \succ_i, \ldots, \succ_n)$, is a dictatorship.

3. Another related concept is the following.

Let $Ch(\succ_1, \ldots, \succ_n)$ be a function that assigns a choice function to every profile of orderings on $X$. We say that $Ch$ satisfies *unanimity* if for every $(\succ_1, \ldots, \succ_n)$ and for any $x, y \in A$, if $y \succ_i x$ for all $i$ then, $x \notin Ch(\succ_1, \ldots, \succ_n)(A)$.

We say that $Ch$ is *invariant to the procedure* if, for every profile $(\succ_1 , \ldots, \succ_n)$ and for every choice set $A$, the following two "approaches" lead to the same outcome:

a. Partition $A$ into two sets $A'$ and $A''$. Choose an element from $A'$ and an element from $A''$ and then choose one element from the two choices.
b. Choose an element from the unpartitioned set $A$.

Dutta, Jackson, and Le Breton (2001) show that only dictatorships satisfy both unanimity and invariance to the procedure.

## Bibliographic Notes

*Recommended readings*: Kreps 1990, chapter 5; Mas-Colell et al. 1995, chapter 21.

This lecture focuses mainly on Arrow's Impossibility Theorem, one of the most famous results in economics, proved by Arrow in his Ph.D. dissertation and published in 1951 (see the classic book Arrow 1963). Social choice theory is beautifully introduced in Sen (1970). The proof brought here is one of many for Arrow's Impossibility Theorem (see Kelly 1988). Reny (2001) provides an elementary proof that demonstrates the strong logical link between Arrow's theorem and the Gibbard-Satterthwaite theorem.

# Problem Set 10

**Problem 1.** (*Moderately difficult.* Based on May 1952.)
Assume that the set of social alternatives, $X$, includes only two alternatives. Define a social welfare function to be a function that attaches a preference to any profile of preferences (allow indifference for the SWF and the individuals' preference relations). Consider the following axioms:

- *Anonymity* If $\sigma$ is a permutation of $N$ and if $p = \{ \succsim_i \}_{i \in N}$ and $p' = \{ \succsim'_i \}_{i \in N}$ are two profiles of preferences on $X$ so that $\succsim'_{\sigma(i)} = \succsim_i$, then $\succsim (p) = \succsim (p')$.
- *Neutrality* For any preference $\succsim_i$ define $(-\succsim_i)$ as the preference satisfying $x(-\succsim_i)y$ iff $y \succsim_i x$. Then $\succsim (\{-\succsim_i\}_{i \in N}) = -\succsim (\{\succsim_i\}_{i \in N})$.
- *Positive Responsiveness* If the profile $\{ \succsim'_i \}_{i \in N}$ is identical to $\{ \succsim_i \}_{i \in N}$ with the exception that for one individual $j$ either ($x \sim_j y$ and $x \succ'_j y$) or ($y \succ_j x$ and $x \sim'_j y$) and if $x \succsim y$ then $x \succ' y$.

a. Interpret the axioms.
b. Does anonymity imply non-dictatorship?
c. Show that the majority rule satisfies all axioms.
d. Prove May's theorem by which the majority rule is the only SWF satisfying the above axioms.
e. Are the above three axioms independent?

**Problem 2.** (*Moderately difficult*)
$N$ individuals choose a single object from among a set $X$. We are interested in functions that aggregate the individuals' recommendations (*not preferences*, just recommendations!) into a social decision (i.e., $F : X^N \to X$).

Discuss the following axioms:

- *Par*: If all individuals recommend $x^*$ then the society chooses $x^*$.
- *I*: If the same individuals support an alternative $x \in X$ in two profiles of recommendations, then $x$ is chosen in one profile if and only if it chosen in the other.

a. Show that if $X$ includes at least three elements, then the only aggregation method that satisfies $P$ and $I$ is a dictatorship.
b. Show the necessity of the three conditions $P$, $I$, and $|X| \geq 3$ for this conclusion.

**Problem 3.** (*Easy*)
Assume that the set of alternatives, $X$, is the interval $[0,1]$ and that each individual's preference is *single-peaked*, i.e., for each $i$ there is an alternative $a_i^*$ such that if $a_i^* \geq b > c$ or $c > b \geq a_i^*$, then $b \succ_i c$.

a. Provide an interpretation of single-peaked preferences.
b. Show that for any odd $n$, if we restrict the domain of preferences to single-peaked preferences, then the majority rule induces a "well-behaved" SWF.

**Problem 4.** (*Moderately difficult.* Based on Kasher and Rubinstein 1997.)
Who is an economist? Departments of economics are often sharply divided over this question. Investigate the approach according to which the determination of who is an economist is treated as an aggregation of the views held by department members on this question.

Let $N = \{1, \ldots, n\}$ be a group of individuals ($n \geq 3$). Each $i \in N$ "submits" a set $E_i$, a *proper* subset of $N$, which is interpreted as the set of "real economists" in his view. An aggregation method $F$ is a function that assigns a *proper* subset of $N$ to each profile $(E_i)_{i=1,\ldots,n}$ of proper subsets of $N$. $F(E_1, \ldots, E_n)$ is interpreted as the set of all members of $N$ who are considered by the group to be economists. (Note that we require that all opinions be proper subsets of $N$.)

Consider the following axioms on $F$:

- *Consensus* If $j \in E_i$ for all $i \in N$, then $j \in F(E_1, \ldots, E_n)$.
- *Independence* If $(E_1, \ldots, E_n)$ and $(G_1, \ldots, G_n)$ are two profiles of views so that for all $i \in N$, $[j \in E_i$ if and only if $j \in F(G_1, \ldots, G_n)]$.

a. Interpret the two axioms.
b. Find one aggregation method that satisfies Consensus but not Independence and one that satisfies Independence but not Consensus.
c. (*Difficult*) Provide a proof similar to that of Arrows' Impossibility Theorem of the claim that the only aggregation methods that satisfy the above two axioms are those for which there is a member $i^*$ such that $F(E_1, \ldots, E_n) \equiv E_{i^*}$.

# Review Problems

The following is a collection of questions I have given in exams during the last few years.

**Problem 1** (Princeton 2002)

Consider a consumer with a preference relation in a world with two goods, $X$ (an aggregated consumption good) and $M$ ("membership in a club," for example), which can be consumed or not. In other words, the consumption of $X$ can be any nonnegative real number, while the consumption of $M$ must be either 0 or 1.

Assume that consumer preferences are strictly monotonic, continuous, and satisfy the following property:

> *Property E*: For every $x$ there is $y$ such that $(y, 0) \succ (x, 1)$ (that is, there is always some amount of money that can compensate for the loss of membership).

1. Show that any consumer's preference relation can be represented by a utility function of the type
$$u(x, m) = \begin{cases} x & \textit{if} \quad m = 0 \\ x + g(x) & \textit{if} \quad m = 1 \end{cases} .$$
2. (Less easy) Show that the consumer's preference relation can also be represented by a utility function of the type
$$u(x, m) = \begin{cases} f(x) & \textit{if} \quad m = 0 \\ f(x) + v & \textit{if} \quad m = 1 \end{cases} .$$
3. Explain why continuity and strong monotonicity (without property E) are not sufficient for (1).
4. Calculate the consumer's demand function.
5. Taking the utility function to be of the form described in (1), derive the consumer's indirect utility function. For the case where the function $g$ is differentiable, verify the Roy equality with respect to commodity $M$.

**Problem 2** (Princeton 2001)

A consumer has to make his decision *before* he is informed whether a certain event, which is expected with probability $\alpha$, happened or not. He assigns a vNM utility $v(x)$ to the consumption of the bundle $x$ in case the event occurs, and a vNM utility $w(x)$ to the consumption of $x$ should the event not occur. The consumer maximizes his expected utility. Both $v$ and $w$ satisfy the standard assumptions about the consumer. Assume also that $v$ and $w$ are concave.

1. Show that the consumer's preference relation is convex.
2. Find a connection between the consumer's indirect utility function and the indirect utility functions derived from $v$ and $w$.
3. A new commodity appears on the market: "A discrete piece of information that tells the consumer whether the event occurred or not." The commodity can be purchased prior to the consumption decision. Use the indirect utility functions to characterize the demand function for the new commodity.

**Problem 3** (Princeton 2001)

1. Define a formal concept for "$\succsim_1$ is closer to $\succsim_0$ than $\succsim_2$."
2. Apply your definition to the class of preference relations represented by $U_1 = tU_2 + (1-t)U_0$, where the function $U_i$ represents $\succsim_i$ $(i = 0, 1, 2)$.
3. Consider the above definition in the consumer context. Denote by $x_k^i(p, w)$ the demand function of $\succsim_i$ for good $k$. Is it true that if $\succsim_1$ is closer to $\succsim_0$ than $\succsim_2$, then $|x_k^1(p, w) - x_k^0(p, w)| \leq |x_k^2(p, w) - x_k^0(p, w)|$ for any commodity $k$ and for every price vector $p$ and wealth level $w$?

**Problem 4** (Princeton 1997)

A decision maker forms preferences over the set $X$ of all possible distributions of a population over two categories (like living in two locations). An element in $X$ is a vector $(x_1, x_2)$ where $x_i \geq 0$ and $x_1 + x_2 = 1$. The decision maker has two considerations in mind:

- He thinks that if $x \succsim y$, then for any $z$, the mixture of $\alpha \in [0, 1]$ of $x$ with $(1-\alpha)$ of $z$ should be at least as good as the mixture of $\alpha$ of $y$ with $(1-\alpha)$ of $z$.

- He is indifferent between a distribution that is fully concentrated in location 1 and one which is fully concentrated in location 2.

1. Show that the only preference relation that is consistent with the two principles is the degenerate indifference relation ($x \sim y$ for any $x, y \in X$).
2. The decision maker claims that you are wrong as his preference relation is represented by a utility function $|x_1 - 1/2|$. Why is he wrong?

**Problem 5** (Princeton 2000. Based on Fishburn and Rubinstein 1982.)

Let $X = \Re^+ \times \{0, 1, 2, \ldots\}$, where $(x, t)$ is interpreted as receiving \$$x$ at time $t$. A preference relation on $X$ has the following properties:

- There is indifference between receiving \$0 at time 0 and receiving \$0 at any other time.
- For any positive amount of money, it is better to receive it as soon as possible.
- Money is desirable.
- The preference between $(x, t)$ and $(y, t+1)$ is independent of $t$.
- Continuity.

1. Define formally the continuity assumption for this context.
2. Show that the preference relation has a utility representation.
3. Verify that the preference relation represented by the utility function $u(x)\delta^t$ (with $\delta < 1$ and $u$ continuous and increasing) satisfies the above properties.
4. Formulize a concept "one preference relation is more impatient than another."
5. Discuss the claim that preferences represented by $u_1(x)\delta_1^t$ are more impatient than preferences represented by $u_2(x)\delta_2^t$ if and only if $\delta_1 < \delta_2$.

**Problem 6** (Tel Aviv 2003)

Consider the following consumer problem. There are two goods, 1 and 2. The consumer has a certain endowment. Before the consumer are two "exchange functions": he can exchange $x$ units of good 1 for $f(x)$ units of good 2, or he can exchange $y$ units of good 2 for $g(y)$ units of good 1. Assume the consumer can only make one exchange.

1. Show that if the exchange functions are continuous and the consumer's preference relation satisfies monotonicity and continuity, then a solution to the consumer problem exists.
2. Explain why strong convexity of the preference relation is not sufficient to guarantee a unique solution if the functions $f$ and $g$ are increasing and convex.
3. What does the statement "the function $f$ is increasing and convex" mean?
4. Suppose both functions $f$ and $g$ are differentiable and concave and that the product of their derivatives at point 0 is 1. Suppose also that the preference relation is strongly convex. Show that under these conditions, the agent will not find two different exchanges, one exchanging good 1 for good 2, and one exchanging good 2 for good 1, optimal.
5. Now assume $f(x) = ax$ and $g(y) = by$. Explain this assumption. Find a condition that will ensure it is not profitable for the consumer to make more than one exchange.

**Problem 7** (Tel Aviv 1999)

Consider a consumer in a world with $K$ goods and preferences satisfying the standard assumptions regarding the consumer. At the start of trade, the consumer is endowed with a bundle of goods $e$ and he chooses the best bundle from the budget set $B(p, e) = \{x|px = pe\}$. The consumer's preference over bundles of goods can be represented by a utility function $u$. Define $V(p, e) = max\{u(x)|px = pe\}$.

1. Explain the meaning of the function $V$ and show that $V(tp, e) = V(p, e)$ where $t$ is any positive number.
2. Show that for every bundle $e$, the set of vectors $p$, such that $V(p, e)$ is less than or equal to $V(p^*, e)$, is convex.
3. Fix all prices but $p_i$, and all quantities in the initial bundle but $w_i$. Show that the slope of the indifference curve of $V$ in the two-dimensional space where the parameters on the axes are $p_i$, and $w_i$ is $(x_i(p, w) - w_i)/p_i$ where $x(p, w)$ is the solution to the consumer's problem $B(p, w)$.

**Problem 8** (Tel Aviv 1998)

A consumer with wealth $w = 10$ "must" obtain a book from one of three stores. Denote the prices at each store as $p_1, p_2, p_3$. All prices are below $w$ in the relevant range. The consumer has devised a strategy: he compares the prices at the first two stores and obtains the book from the first store if its price is not greater than the price at the second store. If $p_1 > p_2$, he compares the prices of the second and third stores and obtains the book from the second store if its price is not greater than the price at the third store. He uses the remainder of his wealth to purchase other goods.

1. What is this consumer's "demand function"?
2. Does this consumer satisfy "rational man" assumptions?
3. Consider the function $v(p_1, p_2, p_3) = w - p_{i^*}$, where $i^*$ is the store from which the consumer purchases the book if the prices are $(p_1, p_2, p_3)$. What does this function represent?
4. Explain why $v(\cdot)$ is not monotonically decreasing in $p_i$. Compare with the indirect utility function of the classic consumer model.

**Problem 9** (Tel Aviv 1999)

Tversky and Kahneman (1986) report the following experiment: each participant receives a questionnaire asking him to make two choices, one from $\{a, b\}$ and the second from $\{c, d\}$:

a. A sure profit of \$240.
b. A lottery between a profit of \$1000 with probability 25% and 0 with probability 75%.

c. A sure loss of \$750.
d. A lottery between a loss of \$1000 with probability 75% and 0 with probability 25%.

The participant will receive the sum of the outcomes of the two lotteries he chooses. Seventy-three percent of participants chose the combination *a* and *d*. What do you make of this result?

**Problem 10** (Princeton 2000)

Consider the following social choice problem: a group has $n$ members ($n$ is odd) who must choose from a set containing 3 elements

$\{A, B, L\}$, where $A$ and $B$ are prizes and $L$ is the lottery which yields each of the prizes $A$ and $B$ with equal probability. Each member has a strict preference over the three alternatives that satisfies vNM assumptions. Show that there is a non-dictatorial social welfare function which satisfies the independence of irrelevant alternatives axiom (even the strict version $I^*$) and the Pareto axiom (*Par*). Reconcile this fact with Arrow's Impossibility Theorem.

**Problem 11** (Tel Aviv 2003. Based on Gilboa and Schmeidler 1995.)

An agent must decide whether to do something, $Y$, or not to do it, $N$.

A history is a sequence of results for past events in which the agent chose $Y$; each result is either a success $S$ or a failure $F$. For example, $(S, S, F, F, S)$ is a history with five events in which the action was carried out. Two of them (events 3 and 4) ended in failure while the rest were successful.

The decision rule $D$ is a function that assigns the decision $Y$ or $N$ to every possible history.

Consider the following properties of decision rules:

- $A1$ After every history that contains only successes, the decision rule will dictate $Y$, and after every history that contains only failures, the decision rule will dictate $N$.
- $A2$ If the decision rule dictates a certain action following some history, it will dictate the same action following any history that is derived from the first history by reordering its members. For example, $D(S, F, S, F, S) = D(S, S, F, F, S)$.
- $A3$ If $D(h) = D(h')$, then this will also be the decision following the concatenation of $h$ and $h'$. (Reminder: The concatenation of $h = (F, S)$ and $h' = (S, S, F)$ is $(F, S, S, S, F)$).

1. For every $i = 1, 2, 3$, give an example of a decision rule that does not fulfill property $Ai$ but does fulfill the other two properties.
2. Give an example of a decision rule that fulfills all three properties.
3. (Difficult) Characterize the decision rules that fulfill the three properties.

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