Lecture Notes

Microeconomic Theory

Guoqiang TIAN

Department of Economics

Texas A&M University

College Station, Texas 77843

(gtian@tamu.edu)

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Contents

Chapter 1

Overview of Economics

We first set out some basic terminologies, methodologies, and assumptions used in modern economics in general and this course in particular.

1.1 Nature and Role of Modern Economics

1.1.1 Modern Economics and Economic Theory

- What is economics about?

Economics is a social science that studies economic phenomena and the economic behavior of individuals, firms, government, and other economic units as well as how they make choices so that limited resources are allocated among competing uses.

Because resources are limited, but people's desires are unlimited, we need economics to study this fundamental conflict.

- Four basic questions must be answered by any economic institution:

(1) What goods and services should be produced and in what quantity?

(2) How should the product be produced?

(3) For whom should it be produced and how should it be distributed?

(4) Who makes the decision?

The answers depend on the use of economic institution. There are two basic economic institutions that have been used in reality:

(1) Market economic institution: Most decisions on economic activities are made by individuals, it is mainly a decentralized decision system.

(2) Planning economic institution: Most decisions on economic activities are made by government, it is mainly a centralized decision system.

- ## What is Modern Economics?

The market economy has been proved to be only economic institution so far that can keep an economy with sustainable development, and therefore *modern economics* mainly studies various economic phenomena and behavior under market economic environment by using an analytical approach.

- ## What is Economic Theory?

Every economic theory that can be considered as an axiomatic approach consists of a set of presumptions and conditions, basic framework, and conclusions that are derived from the assumptions and the framework.

In discussing and applying an economic theory to argue some claims, one should pay attention to the assumptions of the economic theory and the applicable range (boundary and limitation) of the theory.

- ## Microeconomic theory

Microeconomic theory aims to model economic activity as an interaction of individual economic agents pursuing their private interests.

1.1.2 The Standard Analytical Framework of Modern Economics

Modern economics developed in last fifty years stands for an analytical method or framework for studying economic behavior and phenomena. As a theoretical analytical framework of modern economics, it consists of three aspects: perspective, reference system (benchmark), and analytical tools. To have a good training in economic theory, one needs to start from these three aspects. To understand various economic theories and arguments, it is also important for people to understand these three aspects:

(1) Perspective: modern economics provides various perspectives or angles of looking at economic issues starting from reality. An economic phenomenon or issue may be very complicated and affected by many factors. The perspective approach can grasp the most essential factors of the issue and take our attention to most key and core characteristics of an issue so that it can avoid those unimportant details. This should be done by making some key and basic assumptions about preferences, technologies, and endowments. From these basic assumptions, one studies effects of various economic mechanisms (institutions) on behavior of agents and economic units, and takes "equilibrium," "efficiency", "information", and "incentives" as focus points. That is, economists study how individuals interact under the drive of self-interested motion of individuals with a given mechanism, reach some equilibria, and evaluate the status at equilibrium. Analyzing economic problem using such a perspective has not only consistence in methodology, but also get surprising (but logic consistent) conclusions.

(2) Reference Systems (Benchmark): modern economics provides various reference systems. For instance, the general equilibrium theory we will study in this course is such a reference system. Other example includes Coase Theorem in property rights theory and economic law, Modigliani-Miller Theorem in corporate finance theory. The importance of a reference system does not relay on whether or not it describes the real world correctly or precisely, but gives a criterion of understanding the real world. Understanding such a role of reference system is useful to clarify two misunderstandings: One is that some people may over-evaluate a theoretical result in a reference system. They do not know in most situations that a theory does not exactly coincide with the reality, but it only provides a benchmark to see how far a reality is from the ideal status given by a reference system. The other misunderstanding is that some people may

under-evaluate a theoretical result in the reference system and think it is not useful because of unrealistic assumptions. They do not know the value of a theoretical result is not that it can directly explain the world, but that it provides a benchmark for developing new theories to explain the world. In fact, the establishment of a reference system is extremely important for any subject, including economics. Everyone could talk something about an economic issue from realty, but the main difference is that a person with systematic training in modern economics have a few reference systems in her mind while a person without a training in modern economics does not so he cannot grasp an essential part of the issue and cannot provide deep analysis and insights.

(3) Analytical Tools: modern economics provides various powerful analytical tools that are actually given by geometrical or mathematical models. Advantages of such tools can help us to analyze complicated economic behavior and phenomena through a simple diagram or mathematical structure in a model. Examples include (1) the demand-supply curve model, (2) Samuelson's overlapping generation model, and (3) the principal-agent model.

1.1.3 Key Assumptions Commonly Used or Preferred in Modern Economics

Economists usually make some of the following key assumptions and conditions when they study economic problems:

(1) Individuals are (bounded) rational: self-interested behavior assumption;

(2) Scarcity of Resources: Individuals confront scarce resources;

(3) Economic freedom: voluntary cooperation and voluntary exchange;

(4) Decentralized decision makings: One prefers to use the way of decentralized decision marking because most economic information is incomplete to the decision marker;

(5) Incentive compatibility of parties: the system or economic mechanism should solve the problem of interest conflicts between individuals or economic units;

(6) Well-defined property rights;

(7) Equity in opportunity;

(8) Allocative efficiency of resources;

Relaxing any of these assumptions may result in different conclusions.

1.1.4 Roles of Mathematics in Modern Economics

Mathematics has become an important tool used in modern economics. Almost every field in modern economics more or less uses mathematics and statistics. Mathematical approach is an approach to economic analysis in which the economists make use of mathematical symbols in the statement of a problem and also draw upon known mathematical theorems to aid in reasoning. It is not hard to understand why mathematical approach has become a dominant approach because the establishment of a reference system and the development of analytical tools need mathematics. Advantages of using mathematics is that (1) the "language" used is more accurate and precise and the descriptions of assumptions are more clear using mathematics, (2) the logic of analysis is more rigorous and it clearly clarifies the boundary and limitation of a statement, (3) it can give a new result that may not be easily obtained through the observation, and (4) it can reduce unnecessary debates and improve or extend existing results.

It should be remarked that, although mathematics is of critical importance in modern economics, economics is not mathematics, rather economics uses mathematics as a tool to model and analyze various economic problems.

1.1.5 Conversion between Economic and Mathematical Languages

A product in economics science is an economic conclusion. The production of an economic conclusion usually takes three stages: Stage 1 (non-mathematical language stage). Produce preliminary outputs –propose economic ideas, intuitions, and conjectures. (2) Stage 2 (mathematical language stage). Produce intermediate outputs – give a formal and rigorous result through mathematical modeling. Stage 3 (non-technical language stage). Produce final outputs – conclusions, insights, and statements that can be understood by non-specialists.

1.1.6 Limitation and Extension of an Economic Theory

When making an economic conclusion and discussing an economic problem, it is very important to notice the boundary, limitation, and applicable range of an economic theory. Every theory is based on some imposed assumptions, and thus it only relatively correct and has its limitation and boundary of suitability. No theory is universe (absolute). One example is the assumption of perfect competition. In reality, no competition is perfect. Real world markets seldom achieve this ideal. The question is then not whether any particular market is perfectly competitive – almost no market is. The appropriate question is to what degree models of perfect competition can generate insights about real-world market. We think this assumption is approximately correct under some situations. Just like frictionless models in physics such as in free falling body movement (no air resistance), ideal gas (molecules do not collide), ideal fluid, can describe some important phenomena in the physical world, the frictionless models of perfect competition generates useful insights in the economic world.

It is often heard that some people claims they topple an existing theory or conclusion, or they say some theory or conclusions have been overthrown when some conditions or assumptions behind the theory or conclusions are criticized. This is usually not a correct way to say it. We can always criticize any existing theory because no assumption can coincides fully with reality or cover everything. So, as long as there is no logic errors or inconsistency in theory, we cannot say the theory is wrong, although we may criticize that it is too limited or not realistic. What we need to do is to weaken or relax those assumptions, and obtain new theories based on old theories. However, we cannot say this new theory topples the old one, but it may be more appropriate to say that the new theory extends the old theory to cover more general situation or deal with a different economic environments.

1.1.7 Distinguish Necessary and Sufficient Conditions for Statements

In discussing an economic issue, it is very important to distinguish: (1) two types of statements: positive analysis and normative analysis, and (2) two types of conditions: necessary and sufficient conditions for a statement to be true.

Some people often confuse the distinction between necessary condition and sufficient condition when they give their claims, and get wrong conclusions. For instance, it is often heard that the market institution should not used by giving examples that some countries are market economies but are still poor. The reason they get the wrong conclusion that the market mechanism should not be used is because they did not realize the adoption of a market mechanism is just a necessary condition for a country to be rich, but is not a sufficient condition. Becoming a rich country also depends on other factors such as political system, social infrastructures, and culture. So far, no example of a country can be found that it is rich in a long run, but is not a market economy.

1.2 Partial Equilibrium Model for Competitive Markets

The consumer theory and producer theory study maximizing behavior of consumers and producers by taking market prices as given. We begin our study of how the competitive market prices are determined by the actions of the individual agents.

1.2.1 Assumptions on Competitive Market

The competitive markets is based on the following assumptions:

(1) Large number of buyers and sellers — price-taking behavior

(2) Unrestricted mobility of resources among industries: no artificial barrier or impediment to entry or to exit from market.

(3) Homogeneous product: All the firms in an industry produce an identical production in the consumers' eyes.

(4) Passion of all relevant information (all relevant information are common knowledge): Firms and consumers have all the information necessary to make the correct economic decisions.

1.2.2 The Role of Prices

The key insight of Adam Smith's Wealth of Nations is simple: if an exchange between two parties is voluntary, it will not take place unless both believe they will benefit from it. How is this also true for any number of parties and for production case? The price system is the mechanism that performs this task very well without central direction.

Prices perform three functions in organizing economic activities in a free market economy:

(1)They transmit information about production and consumption. The price system transmit only the important information and only to the people who need to know. Thus, it transmits information in an efficiently way.

(2) They provide right incentives. One of the beauties of a free price system is that the prices that bring the information also provide an incentive to react on the information not only about the demand for output but also about the most efficient way to produce a product. They provides incentives to adopt those methods of production that are least costly and thereby use available resources for the most highly valued purposes.

(3) They determine the distribution of income. They determine who gets how much of the product. In general, one cannot use prices to transmit information and provide an incentive to act that information without using prices to affect the distribution of income. If what a person gets does not depend on the price he receives for the services of this resources, what incentive does he have to seek out information on prices or to act on the basis of that information?

1.2.3 The Competitive Firm's Profit Maximization Problem

Since the competitive firm must take the market price as given, its profit maximization problem is simple. The firm only needs to choose output level y so as to solve

$$\max_y py - c(y) \tag{1.1}$$

where y is the output produced by the firm, p is the price of the product, and $c(y)$ is the cost function of production.

The first-order condition (in short, FOC) for interior solution gives:

$$p = c'(y) \equiv MC(y). \tag{1.2}$$

The first order condition becomes a sufficient condition if the second-order condition (in short, SOC) is satisfied

$$c''(y) > 0. \tag{1.3}$$

By $p = c'(y(p))$, we have

$$1 = c''(y(p))y'(p) \tag{1.4}$$

and thus

$$y'(p) > 0, \tag{1.5}$$

which means the law of supply holds.

For the short-run (in short, SR) case,

$$c(y) = c_v(y) + F \tag{1.6}$$

The firm should produce if

$$py(p) - c_v(y) - F \geqq -F, \tag{1.7}$$

and thus we have

$$p \geqq \frac{c_v(y(p))}{y(p)} \equiv AVC. \tag{1.8}$$

That is, the necessary condition for the firm to produce a positive amount of output is that the price is greater than or equal to the average variable cost.

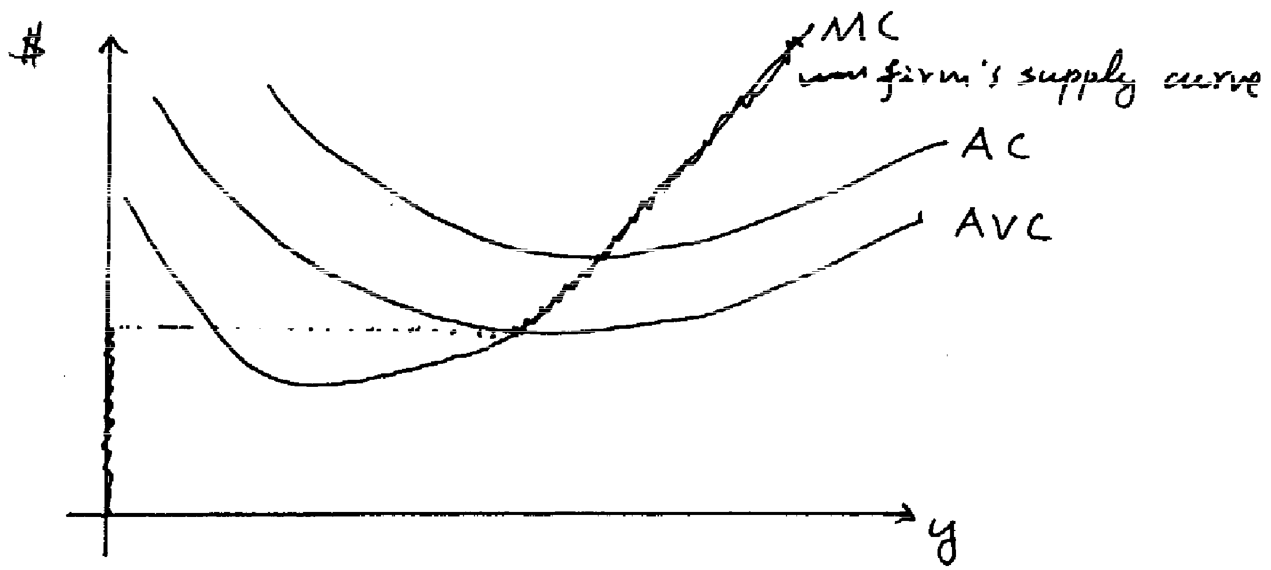

Figure 1.1: Firm's supply curve, and AC, AVC, and MC Curves

The industry supply function is simply the sum of all individuals' supply functions so that it is given by

$$\hat{y}(p) = \sum_{j=1}^{J} y_j(p) \tag{1.9}$$

where $y_i(p)$ is the supply function of firm j for $j = 1, \ldots, J$. Since each firm chooses a level of output where price equals marginal cost, each firm that produces a positive amount of output must have the same marginal cost. The industry supply function measures the relationship between industry output and the common cost of producing this output.

The aggregate (industry) demand function measures the total output demanded at any price which is given by

$$\hat{x}(p) = \sum_{i=1}^{n} x_i(p) \tag{1.10}$$

where $x_i(p)$ is the demand function of consumer i for $1 = 1, \ldots, n$.

1.2.4 Partial Market Equilibrium

A *partial equilibrium price* p^* is a price where the aggregate quantity demanded equals the aggregate quantity supplied. That is, it is the solution of the following equation:

$$\sum_{i=1}^{n} x_i(p) = \sum_{j=1}^{J} y_j(p) \tag{1.11}$$

Example 1.2.1 $\hat{x}(p) = a - bp$ and $c(y) = y^2 + 1$. Since $MC(y) = 2y$, we have

$$y = \frac{p}{2} \tag{1.12}$$

and thus the industry supply function is

$$\hat{y}(p) = \frac{Jp}{2} \tag{1.13}$$

Setting $a - bp = \frac{Jp}{2}$, we have

$$p^* = \frac{a}{b + J/2}. \tag{1.14}$$

Now for general case of $D(p)$ and $S(p)$, what happens about the equilibrium price if the number of firms increases? From

$$D(p(J)) = Jy(p(J))$$

we have

$$D'(p(J))p'(J) = y(p) + Jy'(p(J))p'(J)$$

and thus

$$p'(J) = \frac{y(p)}{X'(p) - Jy'(p)} < 0,$$

which means the equilibrium price decreases when the number of firms increases.

1.2.5 Entry and Long-Run Equilibrium

In the model of entry or exit, the equilibrium number of firms is the largest number of firms that can break even so the price must be chosen to minimum price.

Example 1.2.2 $c(y) = y^2 + 1$. The break-even level of output can be found by setting

$$AC(y) = MC(y)$$

so that $y = 1$, and $p = MC(y) = 2$.

Suppose the demand is linear: $X(p) = a - bp$. Then, the equilibrium price will be the smallest p^* that satisfies the conditions

$$p^* = \frac{a}{b + J/2} \geqq 2.$$

As J increases, the equilibrium price must be closer and closer to 2.

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Part I

General Equilibrium Theory and Social Welfare

Part I is devoted to an examination of competitive market economies from a general equilibrium perspective at which all prices are variable and equilibrium requires that all markets clear.

The content of Part I is organized into three chapters. Chapters 2 and 3 constitute the heart of the general equilibrium theory. Chapter 2 presents the form structure of the equilibrium model, introduces the notion of competitive equilibrium (or called Walrasian equilibrium). The emphasis is on positive properties of the competitive equilibrium. We will discuss the existence, uniqueness, and stability of a competitive equilibrium. We will also discuss a more general setting of equilibrium analysis, namely the abstract economy which includes the general equilibrium model as a special case. Chapter 3 discusses the normative properties of the competitive equilibrium by introducing the notion of Pareto efficiency. We examine the relationship between the competitive equilibrium and Pareto optimality. The core is concerned with the proof of the two fundamental theorems of welfare economics. Chapter 4 explores extensions of the basic analysis presented in Chapters 2 and 3. Chapter 4 covers a number of topics whose origins lie in normative theory. We will study the important core equivalence theorem that takes the idea of Walrasian equilibria as the limit of noncooperative equilibria as markets grow large, fairness of allocation, and social choice theory.

Chapter 2

Positive Theory of Equilibrium: Existence, Uniqueness, and Stability

2.1 Introduction

The general equilibrium theory considers equilibrium in many markets simultaneously, unlike partial equilibrium theory which considers only one market at a time. Interaction between markets may result in a conclusion that is not obtained in a partial equilibrium framework.

A *General Equilibrium* is defined as a state where the aggregate demand will not excess the aggregate supply for all markets. Thus, equilibrium prices are endogenously determined.

The general equilibrium approach has two central features:

(1) It views the economy as a closed and inter-related system in which we must simultaneously determine the equilibrium values of all variables of interests (consider all markets together).

(2) It aims at reducing the set of variables taken as exogenous to a small number of physical realities.

From a positive viewpoint, the general equilibrium theory is a theory of the determination of equilibrium prices and quantities in a system of perfectly competitive markets. It is often called the Walrasian theory of market from L. Walras (1874).

It is to predict the final consumption and production in the market mechanism.

The general equilibrium theory consists of four components:

1. Economic institutional environment (the fundamentals of the economy): economy that consists of consumption space, preferences, endowments of consumers, and production possibility sets of producers.
2. Economic institutional arrangement: It is the price mechanism in which a price is quoted for every commodity.
3. The behavior assumptions: price taking behavior for consumers and firms, utility maximization and profit maximization.
4. Predicting outcomes: equilibrium analysis: positive analysis such as existence, uniqueness, and stability, and normative analysis such as allocative efficiency of general equilibrium.

Questions to be answered in the general equilibrium theory.

A. The existence and determination of a general equilibrium: What kinds of restrictions on economic environments (consumption sets, endowments, preferences, production sets) would guarantee the existence of a general equilibrium.

B. Uniqueness of a general equilibrium: What kinds of restrictions on economic environments would guarantee a general equilibrium to be unique?

C. Stability of a general equilibrium: What kinds of restrictions economic environments would guarantee us to find a general equilibrium by changing prices, especially rasing the price if excess demand prevails and lowering it if excess supply prevails?

D. Welfare properties of a general equilibrium: What kinds of restrictions on consumption sets, endowments, preferences, production sets would ensure a general equilibrium to be social optimal – Pareto efficient?

2.2 The Structure of General Equilibrium Model

Throughout this notes, subscripts are used to index consumers or firms, and superscripts are used to index goods unless otherwise stated. By an agent, we will mean either a consumer or a producer. As usual, vector inequalities, $\geqq$, $\geq$, and $>$, are defined as follows: Let $a, b \in \mathbb{R}^m$. Then $a \geqq b$ means $a_s \geqq b_s$ for all $s = 1, \ldots, m$; $a \geq b$ means $a \geqq b$ but $a \neq b$; $a > b$ means $a_s > b_s$ for all $s = 1, \ldots, m$.

2.2.1 Economic Environments

The fundamentals of the economy are economic institutional environments that are exogenously given and characterized by the following terms:

n: the number of consumers

N: the set of agents

J: the number of producers (firms)

L: the number of (private) goods

$X_i \in \Re^L$: the consumption space of consumer i, which specifies the boundary of consumptions, collection of all individually feasible consumptions of consumer i. Some components of an element may be negative such as a labor supply;

$\succcurlyeq_i$: preferences ordering (or u_i if a utility function exists) of i;

Remark: $\succcurlyeq_i$ is a preference ordering if it is reflexive ($x_i \succcurlyeq_i x_i$), transitive ($x_i \succcurlyeq_i x_i'$ and $x_i' \succcurlyeq_i x_i''$ implies $x_i \succcurlyeq_i x_i''$), and complete (for any pair x_i and x_i', either $x_i \succcurlyeq_i x_i'$ or $x_i' \succcurlyeq_i x_i$). Notice that it can be represented by a utility function if $\succcurlyeq_i$ are continuous. The existence of general equilibrium can be obtained even when preferences are weakened to be non-complete or non-transitive.

$w_i \in X_i$: endowment of consumer i.

$e_i = (X_i, \succcurlyeq_i, w_i)$: the characteristic of consumer i.

Y_j : production possibility set of firm $j = 1, 2, \ldots, J$, which is the characteristic of producer j.

$y_j \in Y_j$: a production plan, $y_j^l > 0$ means y_j^l is output and $y_j^l < 0$ means y_j^l is input. Most elements of y_j for a firm are zero.

Recall there can be three types of returns about production scales: non-increasing returns to scale (i.e., $y_j \in Y_j$ implies that $\alpha y_j \in Y_j$ for all $\alpha \in [0, 1]$), non-decreasing (i.e., $y_j \in Y_j$ implies that $\alpha y_j \in Y_j$ for all $\alpha \geqq 1$) returns to scale, and constant returns to scale i.e., $y_j \in Y_j$ implies that $\alpha y_j \in Y_j$ for all $\alpha \geqq 0$). In other words, decreasing returns to scale implies any feasible input-output vector can be scaled down; increasing returns to scale implies any feasible input-output vector can be scaled up, constant returns to scale implies the production set is the conjunction of increasing returns and decreasing returns. Geometrically, it is a cone.

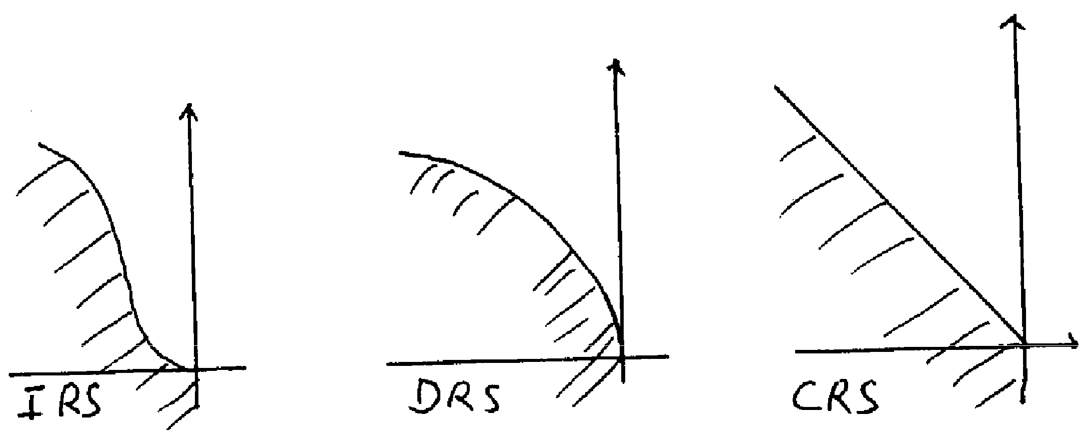

Figure 2.1: Various Returns to Scale: IRS, DRS, and CRS

$e = (\{X_i, \succcurlyeq, w_i\}, \{Y_j\})$: an economy, or called an economic environment.

$X = X_1 \times X_2 \times \ldots \times X_n$: consumption space.

$Y = Y_1 \times Y_2 \times \ldots \times Y_J$: production space.

2.2.2 Institutional Arrangement: the Private Market Mechanism

$p = (p^1, p^2, \dots, p^L) \in \Re^L_+$: a price vector;

px_i: the expenditure of consumer i for $i = 1, \dots, n$;

py_j: the profit of firm J for $j = 1, \dots, J$;

pw_i: the value of endowment of consumer i for $i = 1, \dots, n$;

$\theta_{ij} \in \Re_+$: the profit share of consumer i from firm j, which specifies ownership (property rights) structures, so that $\sum_{i=1}^n \theta_{ij} = 1$ for $j = 1, 2, \dots, J$, and $i = 1, \dots, n$;

$\sum_{j=1}^J \theta_{ij} py_j$ = the total profit dividend received by consumer i for $i = 1, \dots, n$.

For $i = 1, 2, \dots, n$, consumer i's budget constraint is given by

$$px_i \leqq pw_i + \sum_{j=1}^J \theta_{ij} py_j \tag{2.1}$$

and the budget set is given by

$$B_i(p) = \{x_i \in X_i : px_i \leqq pw_i + \sum_{j=1}^J \theta_{ij} py_j\}. \tag{2.2}$$

A private ownership economy then is referred to

$$e = (e_1, e_2, \dots, e_n, \{Y_j\}_{j=1}^n, \{\theta_{ij}\}). \tag{2.3}$$

The set of all such private ownership economies are denoted by E.

2.2.3 Individual Behavior Assumptions:

(1) Perfect Competitive Markets: Every player is a price-taker.

(2) Utility maximization: Every consumer maximizes his preferences subject to $B_i(p)$. That is,

$$\max_{x_i} u_i(x_i) \tag{2.4}$$

s.t.

$$px_i \leqq pw_i + \sum_{j=1}^J \theta_{ij} py_j \tag{2.5}$$

(3) Profit maximization: Every firm maximizes its profit in Y_j. That is,

$$\max_{y_j \in Y_j} p y_j \tag{2.6}$$

for $j = 1, \ldots, J$.

2.2.4 Competitive Equilibrium

Before defining the notion of competitive equilibrium, we first give some notions on allocations which identify the set of possible outcomes in economy e. For notational convenience, "$\hat{a}$" will be used throughout the notes to denote the sum of vectors a_i, i.e., $\hat{a} := \sum a_l$.

Allocation:

An *allocation* (x, y) is a specification of consumption vector $x = (x_1, \ldots, x_n)$ and production vector $y = (y_1, \ldots, y_J)$.

An allocation (x, y) is *individually feasible* if $x_i \in X_i$ for all $i \in N$, $y_j \in Y_j$ for all $j = 1, \ldots, J$.

An allocation is weakly balanced

$$\hat{x} \leqq \hat{y} + \hat{w} \tag{2.7}$$

or specifically

$$\sum_{i=1}^{n} x_i \leqq \sum_{j=1}^{J} y_j + \sum_{i=1}^{n} w_i \tag{2.8}$$

When inequality holds with equality, the allocation is called balanced or attainable.

An allocation (x, y) is feasible if it is both individually feasible and (weakly) balanced.

Thus, an economic allocation is feasible if the total amount of each good consumed does not exceed the total amount available from both the initial endowment and production. Denote by $A = \{(x, y) \in X \times Y : \hat{x} \leqq \hat{y} + \hat{w}\}$ by the set of all feasible allocations.

Aggregation:

$\hat{x} = \sum_{i=1}^{n} x_i$: aggregation of consumption;

$\hat{y} = \sum_{j=1}^{J} y_j$: aggregation of production;

$\hat{w} = \sum_{i=1}^{n} w_i$: aggregation of endowments;

Now we define the notion of competitive equilibrium.

Definition 2.2.1 (Competitive Equilibrium or also called Walrasian Equilibrium) Given a private ownership economy, $e = (e_1, \ldots, e_n, \{Y_j\}, \{\theta_{ij}\})$, an allocation $(x, y) \in$

$X \times Y$ and a price vector $p \in \Re^L_+$ consist of a competitive equilibrium if the following conditions are satisfied

(i) Utility maximization: $x_i \succcurlyeq_i \; x'_i$ for all $x'_i \in B_i(p)$ and $x_i \in B_i(p)$ for $i = 1, \ldots, n$.

(ii) Profit maximization: $py_j \geqq py'_j$ for $y'_j \in Y_j$.

(iii) Market Clear Condition: $\hat{x} \leqq \hat{w} + \hat{y}$.

Denote

$x_i(p) = \{x_i \in B_i(p) : x_i \in B_i(p) \text{ and } x_i \succcurlyeq_i x'_i \text{ for all } x'_i \in B_i(p)\}$: the demand correspondence of consumer i under utility maximization; it is called the demand function of consumer i if it is a single-valued function.

$y_j(p) = \{y_j \in Y_j : py_j \geqq py'_j \text{ for all } y'_j \in Y_j\}$: the supply correspondence of the firm j; it is called the supply function of firm j if it is a single-valued function.

$\hat{x}(p) = \sum_{i=1}^n x_i(p)$: the aggregate demand correspondence.

$\hat{y}(p) = \sum_{j=1}^J y_j(p)$: the aggregate supply correspondence.

$\hat{z}(p) = \hat{x}(p) - \hat{w} - \hat{y}(p)$: aggregate excess demand correspondence.

An equivalent definition of competitive equilibrium then is that a price vector $p^* \in \Re^L_+$ is a competitive equilibrium price if there exists $\hat{z} \in \hat{z}(p^*)$ such that $\hat{z} \leqq 0$.

If $\hat{z}(p)$ is a single-valued, $\hat{z}(p^*) \leqq 0$ is a competitive equilibrium.

2.3 Some Examples of GE Models: Graphical Treatment

In most economies, there are three types of economic activities: production, consumption, and exchanges. Before formally stating the existence results on competitive equilibrium, we will first give two simple examples of general equilibrium models: exchange economies and a production economy with only one consumer and one firm. These examples introduce some of the questions, concepts, and common techniques that will occupy us for the rest of this part.

2.3.1 Pure Exchange Economies

A pure exchange economy is an economy in which there is no production. This is a special case of general economy. In this case, economic activities only consist of trading and consumption.

The aggregate excess demand correspondence becomes $\hat{z}(p) = \hat{x}(p) - \hat{w}$ so that we can define the individual excess demand by $z_i(p) = x_i(p) - w_i$ for this special case.

The simplest exchange economy with the possibility of mutual benefit exchange is the exchange economy with two commodities and two consumers. As it turns out, this case is amenable to analysis by an extremely handy graphical device known as the Edgeworth box.

- **Edgeworth Box**:

Consider an exchange economy with two goods (x_1, x_2) and two persons. The total endowment is $\hat{w} = w_1 + w_2$. For example, if $w_1 = (1, 2), w_2 = (3, 1)$, then the total endowment is: $\hat{w} = (4, 3)$. Note that the point, denoted by w in the Edgeworth Box, can be used to represent the initial endowments of two persons.

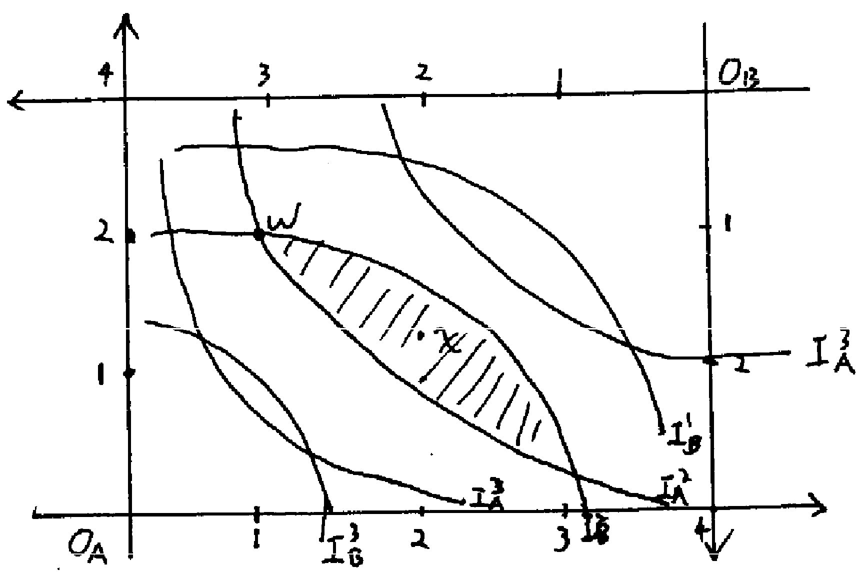

Figure 2.2: Edgeworth Box in which $w_1 = (1, 2)$ and $w_2 = (3, 1)$

Advantage of the Edgeworth Box is that it gives all the possible (balanced) trading points. That is,

$$x_1 + x_2 = w_1 + w_2 \tag{2.9}$$

for all points $x = (x_1, x_2)$ in the box, where $x_1 = (x_1^1, x_1^2)$ and $x_2 = (x_2^1, x_2^2)$. Thus, every point in the Edgeworth Box stands for an attainable allocation so that $x_1 + x_2 = w_1 + w_2$.

The shaded lens (portion) of the box in the above figure represents all the trading points that make both persons better off. Beyond the box, any point is not feasible.

Which point in the Edgeworth Box can be a competitive equilibrium?

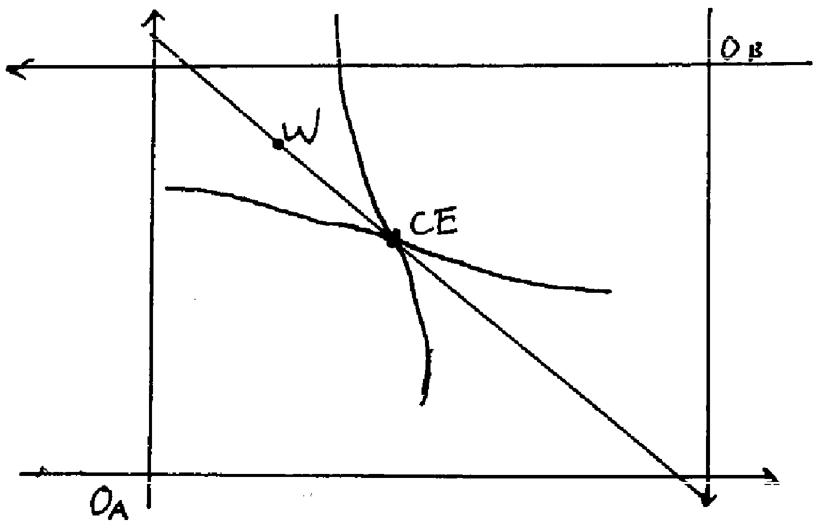

Figure 2.3: In the Edgeworth Box, the point CE is a competitive equilibrium.

In the box, one person's budget line is also the budget line of the other person. They share the same budget line in the box.

Figure 2.4 below shows the market adjustment process to a competitive equilibrium. Originally, at price p, both persons want more good 2. This implies that the price of good 1, p^1, is too high so that consumers do not consume the total amounts of the good so that there is a surplus for x_1 and there is an excess demand for x_2, that is, $x_1^1 + x_2^1 < w_1^1 + w_2^1$ and $x_1^2 + x_2^2 > w_1^2 + w_2^2$. Thus, the market will adjust itself by decreasing p^1 to $p^{1\prime}$. As a result, the budget line will become flatter and flatter till it reaches the equilibrium where the aggregate demand equals the aggregate supply. In this interior equilibrium case, two indifference curves are tangent each other at a point that is on the budget line so that the marginal rates of substitutions for the two persons are the same that is equal to the price ratio.

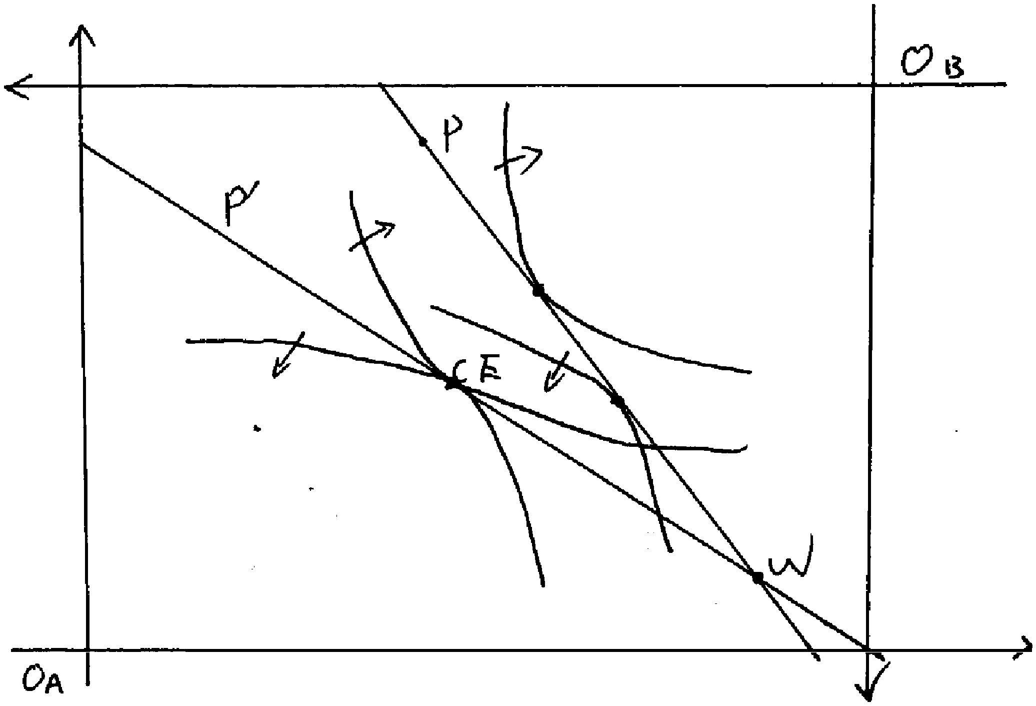

Figure 2.4: This figure shows the market adjustment process

What happens when indifference curves are linear?

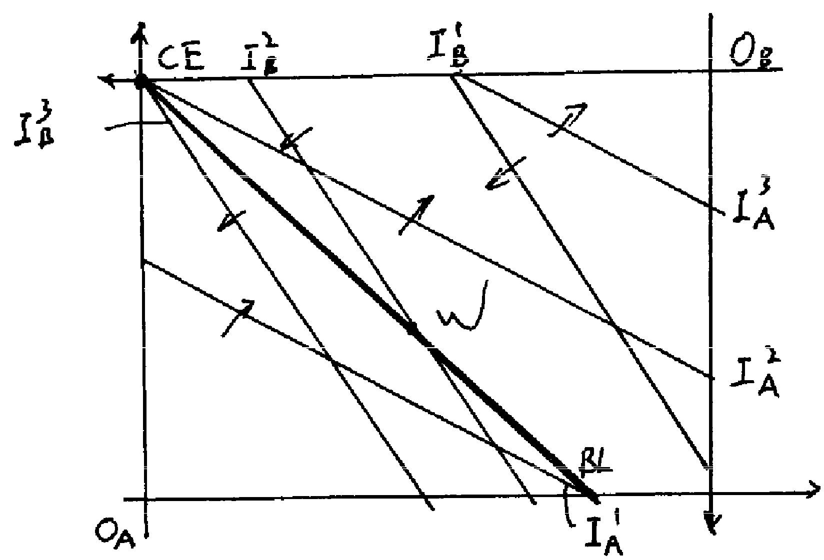

Figure 2.5: A competitive equilibrium may still exist even if two persons' indifference curves do not intersect.

In this case, there is no tangent point as long as the slopes of the indifference curves of the two persons are not the same. Even so, there still exists a competitive equilibrium although the marginal rates of substitutions for the two persons are not the same.

Offer Curve: the locus of the optimal consumptions for the two goods when price varies. Note that, it consists of tangent points of the indifference curves and budget lines when price varies.

The offer curves of two persons are given in the following figure:

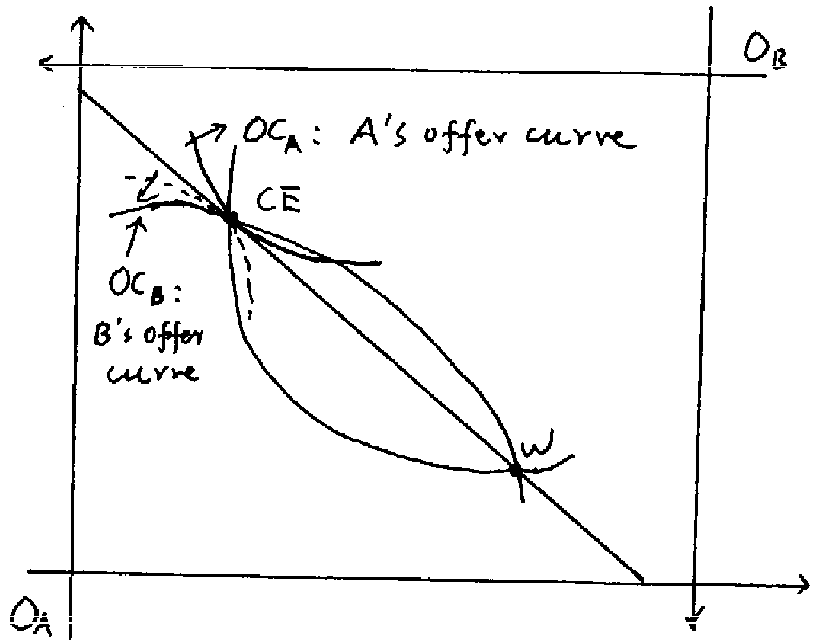

Figure 2.6: The CE is characterized by the intersection of two persons' offer curves.

The intersection of the two offer curves of the consumers can be used to check if there is a competitive equilibrium. By the above diagram, one can see that the one intersection of two consumers' offer curves is always given at the endowment w. If there is another intersection of two consumers' offer curves rather than the one at the endowment w, the intersection point must be the competitive equilibrium.

To enhance the understanding about the competitive equilibrium, you try to draw the competitive equilibrium in the following situations:

1. There are many equilibrium prices.
2. One person's preference is such that two goods are perfect substitutes (i.e., indifference curves are linear).
3. Preferences of one person are the Leontief-type (perfect complement)
4. One person's preferences are non-convex.

5. One person's preferences are "thick".

6. One person's preferences are convex, but has a satiation point.

Note that a preference relation $\succcurlyeq_i$ is convex if $x \succcurlyeq_i x'$ implies $tx + (1-t)x' \succcurlyeq_i x'$ for all $t \in [0,1]$ and all $x, x' \in X_i$. A preference relation $\succcurlyeq_i$ has a satiation point x if $x \succcurlyeq_i x'$ all $x' \in X_i$.

Cases in which there may be no Walrasian Equilibria:

Case 1. Indifference curves (IC) are not convex. If the two offer curves may not be intersected except for at the endowment points, then there may not exist a competitive equilibrium. This may be true when preferences are not convex.

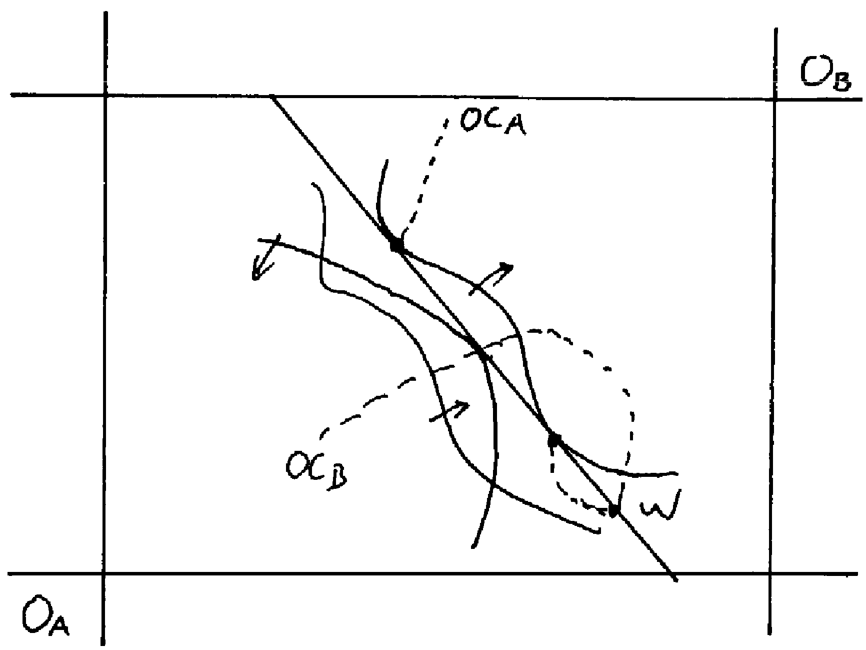

Figure 2.7: A CE may not exist if indifference curves are not convex. In that case, two persons' offer curves do not intersect.

Case 2. The initial endowment may not be an interior point of the consumption space. Consider an exchange economy in which one person's indifference curves put no values on one commodity, say, when Person 2's indifference curves are vertical lines so that $u_2(x_2^1, x_2^2) = x_2^1$. Person 1's utility function is regular one, say, which is given by the Cobb-Douglas utility function. The initial endowments are given by $w_1 = (0,1)$ and $w_2 = (1,0)$. There may not be a competitive equilibrium. Why?

If $p^1/p^2 > 0$, then

$$\begin{cases} x_2^1 = 1 \\ x_1^1 > 0 \end{cases} \tag{2.10}$$

so that $x_1^1(p) + x_2^1(p) > 1 = \hat{w}^1$. Thus, there is no competitive equilibrium.

If $p^1/p^2 = 0$, then $x_2^1 = \infty$ and thus there is not competitive equilibrium. A competitive equilibrium implies that $x_1^1(p) + x_2^1(p) \leqq \hat{w}^1$. However, in both sub-cases, we have $x_1^1(p) + x_2^1(p) > \hat{w}^1$ which violates the feasibility conditions.

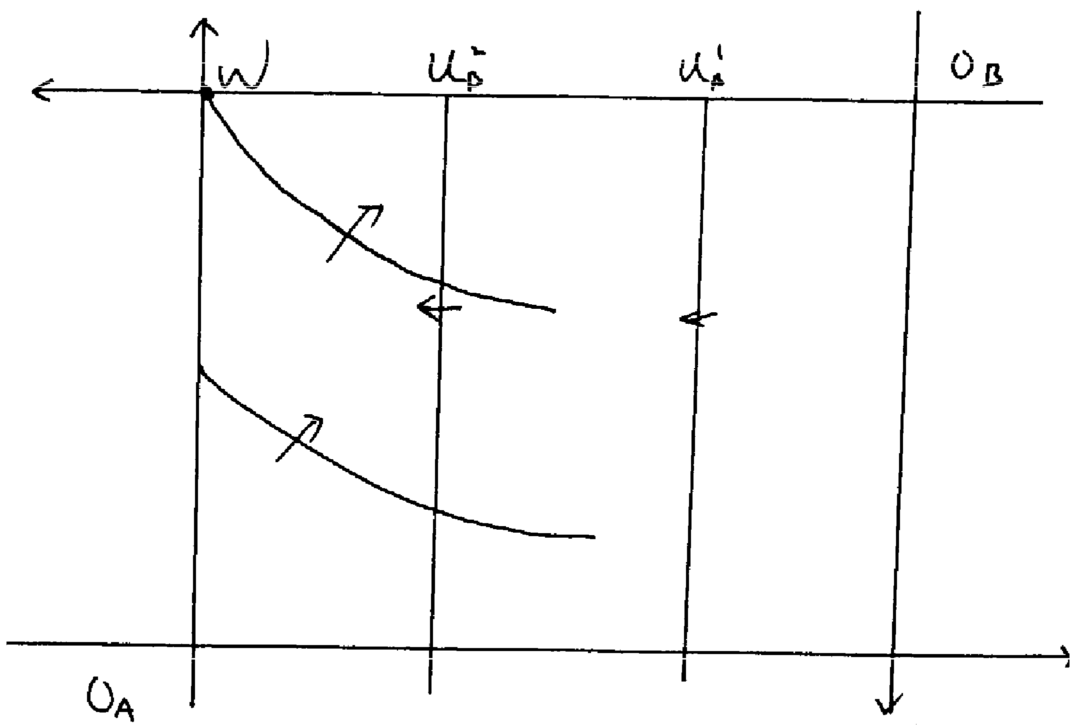

Figure 2.8: A CE may not exist if an endowment is on the boundary.

2.3.2 The One-Consumer and One Producer Economy

Now we introduce the possibility of production. To do so, in the simplest-possible setting in which there are only two price-taking economic agents.

Two agents: one producer so that $J = 1$ and one consumer so that $n = 1$.

Two goods: labor (leisure) and the consumption good produced by the firm.

$w = (\bar{L}, 0)$: the endowment.

$\overline{L}$: the total units of leisure time.

$f(z)$: the production function that is strictly increasing, concave, and differentiable, where z is labor input. To have an interior solution, we assume f satisfies the Inada condition $f'(0) = +\infty$ and $\lim_{z\to 0} f'(z)z = 0$.

(p, w): the price vector of the consumption good and labor.

$\theta = 1$: single person economy.

$u(x^1, x^2)$: is the utility function which is strictly quasi-concave, increasing, and differentiable. To have an interior solution, we assume u satisfies the Inada condition $\frac{\partial u}{\partial x_i}(0) = +\infty$ and $\lim_{x_i\to 0} \frac{\partial u}{\partial x_i} x_i = 0$.

The firm's problem is to choose the labor z so as to solve

$$\max_{z \geqq 0} pf(z) - wz \tag{2.11}$$

FOC:

$$\begin{aligned} pf'(z) &= w \\ \Rightarrow \quad f'(z) &= w/p \\ (MRTS_{z,q} &= \text{Price ratio}) \end{aligned}$$

which means the marginal rate of technique substitution of labor for the consumption good q equals the price ratio of the labor input and the consumption good output.

Let

$q(p, w)$ = the profit maximizing output for the consumption good.

$z(p, w)$ = the profit maximizing input for the labor.

$\pi(p, w)$ = the profit maximizing function.

The consumer's problem is to choose the leisure time and the consumption for the good so as to solve

$$\begin{aligned} &\max_{x^1, x^2} u(x^1, x^2) \\ s.t. \quad &px^2 \leqq w(\overline{L} - x^1) + \pi(p, w) \end{aligned}$$

where x^1 is the leisure and x^2 is the consumption of good.

(FOC:)

$$\frac{\frac{\partial u}{\partial x^1}}{\frac{\partial u}{\partial x^2}} = \frac{w}{p} \tag{2.12}$$

which means the marginal rate of substitution of the leisure consumption for the consumption good q equals the price ratio of the leisure and the consumption good, i.e., $MRS_{x^1x^2} = w/p$.

By (2.11) and (2.12)

$$MRS_{x^1x^2} = \frac{w}{p} = MRTS_{z,q} \tag{2.13}$$

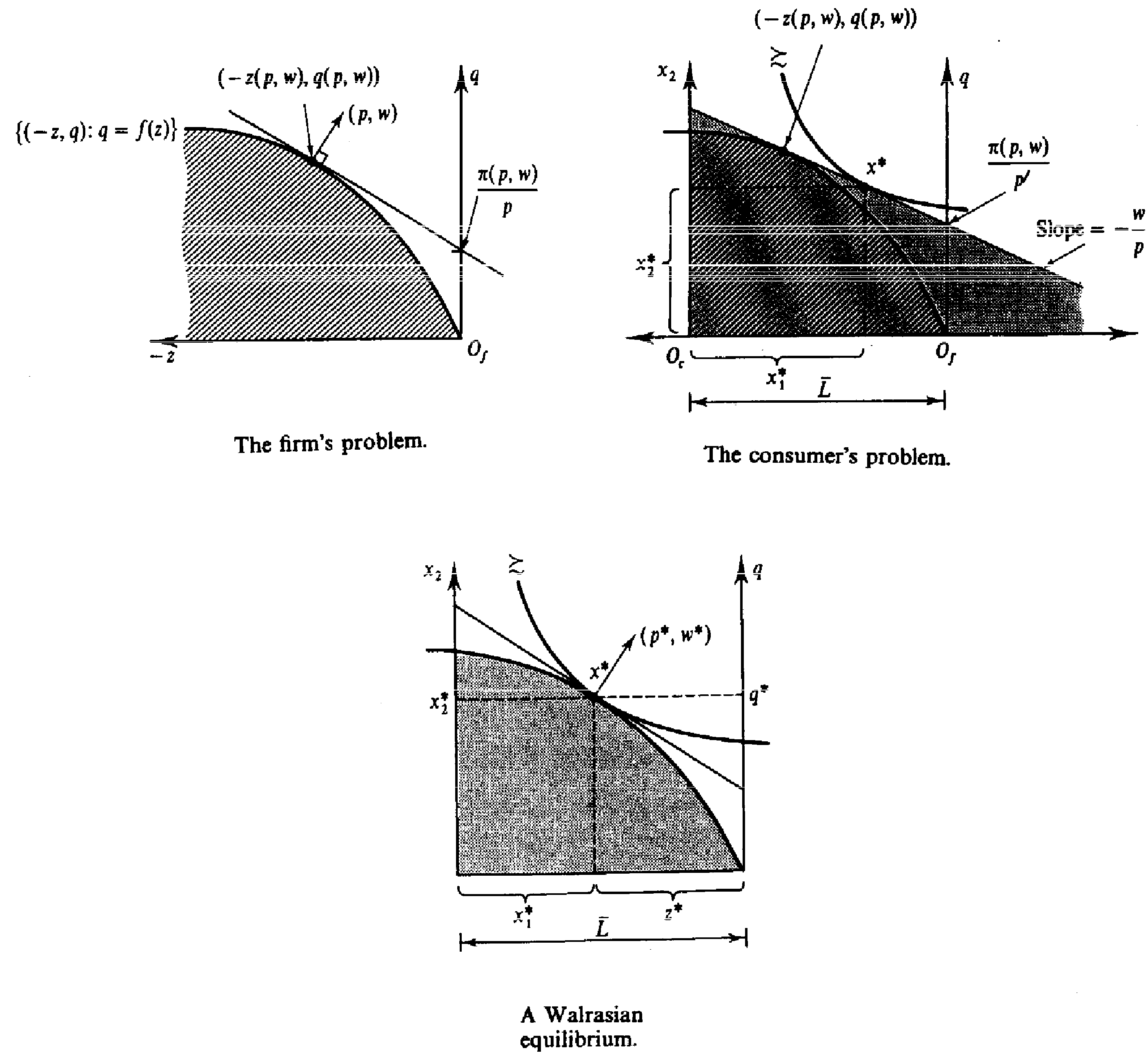

Figure 2.9: Figures for the producer's problem, the consumer problem and the CE.

A competitive equilibrium for this economy involves a price vector (p^*, w^*) at which

$$\begin{aligned} x^2(p^*, w^*) &= q(p^*, w^*); \\ x^1(p^*, w^*) &+ z(p^*, w^*) = \overline{L} \end{aligned}$$

That is, the aggregate demand for the two goods equals the aggregate supply for the two goods. Figure 2.9 shows the problems of firm and consumer, and the competitive equilibrium, respectively.

2.4 The Existence of Competitive Equilibrium

In this section we will examine the existence of competitive equilibrium for the three cases: (1) the single-valued aggregate excess demand function; (2) the aggregate excess demand correspondence; (3) a general class of private ownership production economies. The first two cases are based on excess demand instead of underlying preference orderings and consumption and production sets. There are many ways to prove the existence of general equilibrium. For instance, one can use the Brouwer fixed point theorem approach, KKM lemma approach, and abstract economy approach to show the existence of competitive equilibrium for these three cases. Before giving the existence result on competitive equilibrium, we first provide some notation, definitions and mathematics results used in this section.

2.4.1 Fixed Point Theorems, KKM Lemma, Maximum Theorem, and Separating Hyperplane Theorem

This subsection gives various definitions on continuity of functions and correspondences, fixed point theorems, KKM lemma, maximum theorem, and Separating Hyperplane Theorem which will be used to prove the existence of competitive equilibrium for various economic environments.

Let X and Y be two topological spaces, and let 2^Y be the collection of all subsets of Y. Although most the results given in this subsection is held for general topological vector spaces, for simplicity, we will restrict X and Y to subsets of Euclidian spaces.

Definition 2.4.1 A function $f: X \to \mathbb{R}$ is said to be *continuous* if at point $x_0 \in X$,

$$\lim_{x \to x_0} f(x) = f(x_0),$$

or equivalently, for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x \in X$ satisfying $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| < \epsilon$$

A function $f: X \to \mathbb{R}$ is said to be continuous *on X if f is continuous at every point $x \in X$.*

The so-called upper semi-continuity and lower semi-continuity continuities are weaker than continuity. Even weak conditions on continuity are transfer continuity which characterize many optimization problems and can be found in Tian (1992, 1993, 1994) and Tian and Zhou (1995), and Zhou and Tian (1992).

Definition 2.4.2 A function $f: X \to \mathbb{R}$ is said to be *upper semi-continuous* if at point $x_0 \in X$, we have

$$\limsup_{x \to x_0} f(x) \leqq f(x_0),$$

or equivalently,

$$F(x_0) \equiv \{x \in X : f(x) \geqq f(x_0)\} \text{ is a closed subset of } X$$

or equivalently, for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x \in X$ satisfying $|x - x_0| < \delta$, we have

$$f(x) < f(x_0) + \epsilon$$

A function $f: X \to \mathbb{R}$ is said to be *upper semi-continuous* on X if f is upper semi-continuous at every point $x \in X$.

Definition 2.4.3 A function $f: X \to \mathbb{R}$ is said to be *lower semi-continuous* on X if $-f$ is upper semi-continuous.

It is clear that a function $f: X \to \mathbb{R}$ is continuous on X if and only if it is both upper and lower semi-continuous.

To show the existence of a competitive equilibrium for the continuous aggregate excess demand function, we may use the following fixed-point theorem. The generalization of

Brouwer's fixed theorem can be found in Tian (1991) that gives necessary and sufficient conditions for a function to have a fixed point.

Theorem 2.4.1 (Brouwer's Fixed Theorem) *Let X be a non-empty, compact, and convex subset of $\mathbb{R}^m$. If a function $f : X \to X$ is continuous on X, then f has a fixed point, i.e., there is a point $x^* \in X$ such that $f(x^*) = x^*$.*

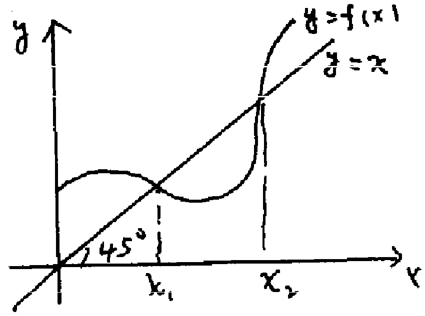

Figure 2.10: Fixed points are given by the intersections of the 45^0 line and the curve of the function.

Example 2.4.1 $f : [0, 1] \to [0, 1]$ is continuous, then f has a fixed point (x). To see this, let

$g(x) = f(x) - x.$

Then, we have

$g(0) = f(0) \geqq 0$

$g(1) = f(1) - 1 \leqq 0.$

From the mean-value theorem, there is a point $x^* \in [0, 1]$ such that $g(x^*) = f(x^*) - x^* = 0$.

When the aggregate excess demand correspondence is not a single-valued function or is not a continuous function, we need the fixed point theorem for correspondence or KKM lemma to prove the existence of competitive equilibrium.

Definition 2.4.4 A correspondence $F : X \to 2^Y$ is *upper hemi-continuous* if the set $\{x \in X : F(x) \subset V\}$ is open in X for every open set subset V of Y.

We also use $F : X \to\to Y$ to denote the mapping $F : X \to 2^Y$ in this lecture notes.

Definition 2.4.5 A correspondence $F : X \to 2^Y$ is said to be *lower hemi-continuous at* x if for any $\{x_k\}$ with $x_k \to x$ and $y \in F(x)$, then there is a sequence $\{y_k\}$ with $y_k \to y$ and $y_n \in F(x_k)$. F is said to be lower hemi-continuous on X if F is lower hemi-continuous for all $x \in X$ or equivalently, if the set $\{x \in X : F(x) \cap V \neq \emptyset\}$ is open in X for every open set subset V of Y.

Definition 2.4.6 A correspondence $F : X \to 2^Y$ is said to be *continuous* if it is both upper hemi-continuous and lower hemi-continuous.

Remark 2.4.1 As with upper (lower) hemi-continuous correspondence, if $f(\cdot)$ is a function then the concept of upper (lower) hemi-continuity as a correspondence and of continuity as a function coincide.

Definition 2.4.7 A correspondence $F : X \to 2^Y$ is said to be *closed* at x if for any $\{x_k\}$ with $x_k \to x$ and $\{y_k\}$ with $y_k \to y$ and $y_n \in F(x_k)$ implies $y \in F(x)$. F is said to be closed if F is closed for all $x \in X$ or equivalently

$$Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\} \text{ is closed.}$$

Remark 2.4.2 If Y is compact and F is closed, then F is upper hemi-continuous.

Definition 2.4.8 A correspondence $F : X \to 2^Y$ said to be *open* if ifs graph

$$Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\} \text{ is open.}$$

Definition 2.4.9 A correspondence $F : X \to 2^Y$ said to have *upper open sections* if $F(x)$ is open for all $x \in X$.

A correspondence $F : X \to 2^Y$ said to have *lower open sections* if its inverse set $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open.

Remark 2.4.3 If a correspondence $F : X \to 2^Y$ has an open graph, then it has upper and lower open sections. If a correspondence $F : X \to 2^Y$ has lower open sections, then it must be lower hemi-continuous.

Theorem 2.4.2 (Kakutani's Fixed Point Theorem) *Let X be a non-empty, compact, and convex subset of $\mathbb{R}^m$. If a correspondence $F : X \to 2^X$ is a upper hemi-continuous correspondence with non-empty compact and convex values on X, then F has a fixed point, i.e., there is a point $x^* \in X$ such that $x^* \in F(x^*)$.*

The Knaster-Kuratowski-Mazurkiewicz (KKM) lemma is quite basic and in some ways more useful than Brouwer's fixed point theorem. The following is a generalized version of KKM lemma due to Ky Fan (1984).

Theorem 2.4.3 (FKKM Theorem) *Let Y be a convex set and $\emptyset \not\subset X \subset Y$. Suppose $F : X \to 2^Y$ is a correspondence such that*

(1) $F(x)$ is closed for all $x \in X$;

(2) $F(x_0)$ is compact for some $x_0 \in X$;

(3) F is FS-convex, i.e, for any $x_1, \ldots, x_m \in X$ and its convex combination $x_\lambda = \sum_{i=1}^m \lambda_i x_i$, we have $x_\lambda \in \cup_{i=1}^m F(x_i)$.

Then $\cap_{x \in X} F(x) \neq \emptyset$.

Here, The term FS is for Fan (1984) and Sonnenschein (1971), who introduced the notion of FS-convexity.

Theorem 2.4.4 (Berg's Maximum Theorem) *Suppose $f(x, a)$ is a continuous function mapping from A to X, and the constraint set $F : A \to\to X$ is a continuous correspondence with non-empty compact-valued values. Then, the optimal valued function (also called marginal function):*

$$M(a) = \max_{x \in F(a)} f(x, a)$$

is a continuous function, and the optimal solution:

$$\phi(a) = \arg \max_{x \in F(a)} f(x, a)$$

is a upper hemi-continuous correspondence.

The various characterization results on Kakutani's Fixed Point Theorem, KKM Lemma, and Maximum Theorem can be found in Tian (1991, 1992, 1994) and Tian and Zhou (1992).

Theorem 2.4.5 (Separating Hyperplane Theorem) *Suppose that $A, B \subset \mathbb{R}^m$ are convex and $A \cap B \neq \emptyset$. Then, there is a vector $p \in \mathbb{R}^m$ with $p \neq 0$, and a value $c \in \mathbb{R}$ such that*

$$px \leqq c \leqq py \quad \forall x \in A \;\&\; y \in B.$$

Furthermore, suppose that $B \subset \mathbb{R}^m$ is convex and closed, $A \subset \mathbb{R}^m$ is convex and compact, and $A \cap B \neq \emptyset$. Then, there is a vector $p \in \mathbb{R}^m$ with $p \neq 0$, and a value $c \in \mathbb{R}$ such that

$$px < c < py \quad \forall x \in A \;\&\; y \in B.$$

2.4.2 The Existence of CE for Aggregate Excess Demand Functions

The simplest case for the existence of a competitive equilibrium is the one when the aggregate excess demand correspondence is a single-valued function. Assume the following properties on the aggregate excess demand correspondence $\hat{z}(p)$.

Function: $\hat{z}(p)$ is a function. It is so if preference orderings are strictly convex.

Continuity: $\hat{z}(p)$ is continuous. Note that if $x_i(p)$ and $y_j(p)$ are continuous, $\hat{z}(p)$ is also continuous.

Walras' Law: for all $p \geqq 0$,

$$p \cdot \hat{z}(p) = 0. \tag{2.14}$$

Homogeneity of $\hat{z}(p)$: it is homogeneous of degree 0 in price $\hat{x}(\lambda p) = \hat{z}(p)$ for any $\lambda > 0$. From this property, we can normalize prices.

Because of homogeneity, for example, we can normalize a price vector as follows:

(1) $p'^l = p^l/p^1 \quad l = 1, 2, \ldots, L$

(2) $p'^l = p^l / \sum_{l=1}^{L} p^l$.

Thus, without loss of generality, we can restrict our attention to the unit simplex:

$$S^{L-1} = \{p \in \Re^L_+ : \sum_{l=1}^{L} p^l = 1\}. \tag{2.15}$$

Then, we have the following theorem on the existence of competitive equilibrium.

Theorem 2.4.6 (The Existence Theorem I)) *For a private ownership economy $e = (\{X_i, w_i, \succcurlyeq_i\}, \{Y_j\}, \{\theta_{ij};\})$, if $\hat{z}(p)$ is a continuous function and satisfies Walras' Law, then there exists a competitive equilibrium, that is, there is $p^* \in \Re^L_+$ such that*

$$\hat{z}(p^*) \leqq 0 \tag{2.16}$$

Proof: Define a continuous function $g: S^{L-1} \to S^{L-1}$ by

$$g^l(p) = \frac{p^l + \max\{0, \hat{z}^l(p)\}}{1 + \sum_{k=1}^{L} \max\{0, \hat{z}^k(p)\}} \tag{2.17}$$

for $l = 1, 2, \ldots, L$.

First note that g is a continuous function since $\max\{f(x), h(x)\}$ is continuous when $f(x)$ and $h(x)$ are continuous.

By Brouwer's fixed point theorem, there exists a price vector p^* such that $g(p^*) = p^*$, i.e.,

$$p^{*l} = \frac{p^{*l} + \max\{0, \hat{z}^l(p^*)\}}{1 + \sum_{k=1}^{L} \max\{0, \hat{z}^k(p^*)\}} \quad l = 1, 2, \ldots, L. \tag{2.18}$$

We want to show p^* is in fact a competitive equilibrium price vector.

Cross multiplying $1 + \sum_{k=1}^{L} \max\{0, \hat{z}^k(p^*)$ on both sides of (2.18), we have

$$p^{*l} \sum_{k=1}^{L} \max\{0, \hat{z}^k(p^*)\} = \max\{0, \hat{z}^l(p^*)\}. \tag{2.19}$$

Then, multiplying the above equation by $\hat{z}^l(p^*)$ and making summation, we have

$$\left[\sum_{l=1}^{L} p^{*l} \hat{z}^l(p^*)\right] \left[\sum_{l=1}^{L} \max\{0, \hat{z}^l(p^*)\}\right] = \sum_{l=1}^{L} \hat{z}^l(p^*) \max\{0, \hat{z}^l(p^*)\}. \tag{2.20}$$

Then, by Walras' Law, we have

$$\sum_{l=1}^{L} \hat{z}^l(p^*) \max\{0, \hat{z}^l(p^*)\} = 0. \tag{2.21}$$

Therefore, each term of the summations is either zero or $(\hat{z}^l(p^*))^2 > 0$. Thus, to have the summation to be zero, we must have each term to be zero. That is, $\hat{z}^l(P^*) \leqq 0$ for $l = 1, \ldots, L$. The proof is completed.

Remark 2.4.4 By Walras' Law, if $p > 0$, and if $(L-1)$ markets are in the equilibrium, the L-th market is also in the equilibrium.

Remark 2.4.5 *Do not confuse a aggregate excess demand function with Walras' Law. Even though Walras' Law holds, we may not have* $\hat{z}(p) \leqq 0$ *for all* p. *Also, if* $\hat{z}(p^*) = 0$ *for some* p^*, *i.e.,* p^* *is a competitive equilibrium price vector, the Walras' Law may not hold unless some types of monotonicity are imposed.*

Fact 1: (free goods). Under Walras' Law, if p^* is a competitive equilibrium price vector and $\hat{z}^l(p^*) < 0$, then $p^{*l} = 0$.

Proof. Suppose not. Then $p^{*l} > 0$. Thus, $p^{*l}\hat{z}^l(p^*) < 0$, and so $p^*\hat{z}(p^*) < 0$, contradicting Walras' Law.

Desirable goods: if $p^l = 0$, $\hat{z}^l(p^*) > 0$.

Fact 2: (Equality of demand and supply). If all goods are desirable and p^* is a competitive equilibrium price vector, then $\hat{z}(p^*) = 0$.

Proof. Suppose not. We have $\hat{z}^l(p^*) < 0$ for some l. Then, by Fact 1, we have $p^{*l} = 0$. Since good l is desirable, we must have $\hat{z}^l(p^{*l}) > 0$, a contradiction.

From the above theorem, Walras' Law is important to prove the existence of a competitive equilibrium. Under which conditions, is Walras' Law held?

- **Conditions for Walras' Law to be true**

When each consumer's budget constraint holds with equality:

$$px_i = pw_i + \sum_{j=1}^{J} \theta_{ij} p y_j$$

for all i, we have

$$\begin{aligned}\sum_{i=1}^{n} px_i(p) &= \sum_{i=1}^{n} pw_i + \sum_{i=1}^{n}\sum_{j=1}^{J} \theta_{ij} p y_j \\ &= \sum_{i=1}^{n} pw_i + \sum_{j=1}^{J} p y_j\end{aligned}$$

which implies that

$$\left[\sum_{i=1}^{n} x_i(p) - \sum_{i=1}^{n} w_i - \sum_{j=1}^{J} p y_j\right] = 0 \tag{2.22}$$

so that

$$p \cdot \hat{z} = 0 \quad (Walras' Law) \tag{2.23}$$

Thus, as long as the budget line holds with equality, Walras' Law must hold.

The above existence theorem on competitive equilibrium is based on the assumptions that the aggregate excess demand correspondence is single-valued and satisfies the Walras's Law. The questions are under what conditions on economic environments a budget constraint holds with equality, and the aggregate excess demand correspondence is single-valued or convex-valued? The following various types of monotonicities and convexities of preferences with the first one strongest and the last one weakest may be used to answer these questions.

- Types of monotonicity conditions

  (1) Strict monotonicity: For any two consumption market bundles ($x \geq x'$) with $x \neq x'$ implies $x \succ_i x'$.

  (2) Monotonicity: if $x > x'$ implies that $x \succ_i x'$.

  (3) Local non-satiation: For any point x and any neighborhood, $N(x)$, there is $x' \in N(x)$ such that $x' \succ_i x$.

  (4) Non-satiation: For any x, there exists x' such that $x' \succ_i x$.

- Types of convexities

  (i) Strict convexity: For any x and x' with $x \succcurlyeq_i x'$ and $x \neq x'$, $x_\lambda \equiv \lambda x + (1 - \lambda)x' \succ_i x'$ for $\lambda \in (0, 1)$.

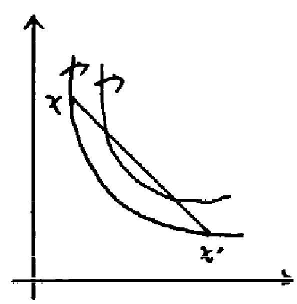

Figure 2.11: Strict convex indifference curves.

  (ii) Convexity: If $x \succ_i x'$, then $x_\lambda = \lambda x + (1 - \lambda)x' \succ_i x'$ for $\lambda \in (0, 1)$.

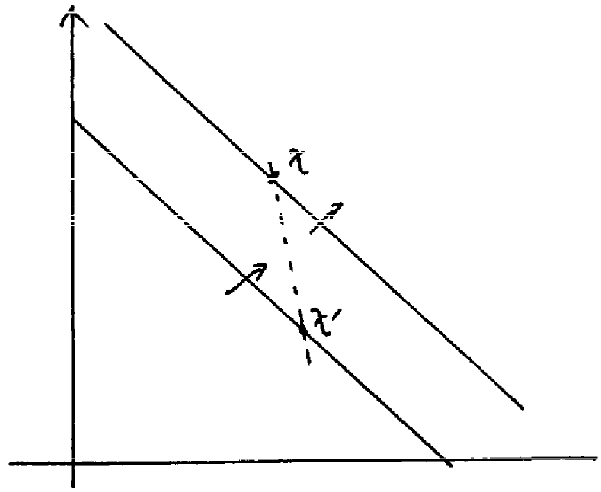

Figure 2.12: Linear indifference curves are convex, but not strict convex.

(iii) Weak convexity: If $x \succcurlyeq_i x'$, then $x_\lambda \succcurlyeq_i x'$.

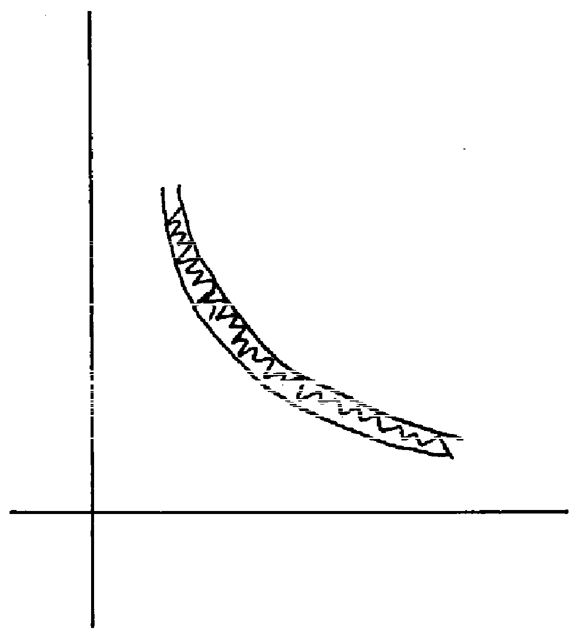

Figure 2.13: "Thick" indifference curves are weakly convex, but not convex.

Each of the above condition implies the next one, the converse may not be true by examining thick indifference curves and linear indifference curves, strictly convex indifference curves as showed in the above figures:

Remark 2.4.6 Monotonicity of preferences can be interpreted as individuals' desires for goods: the more, the better. Local non-satiation means individuals' desires are unlimited.

Remark 2.4.7 The convexity of preferences implies that people want to diversify their consumptions, and thus, convexity can be viewed as the formal expression of basic measure of economic markets for diversification. Note that the strict convexity of $\succ_i$ implies the conventional diminishing marginal rates of substitution (MRS), and weak convexity of $\succcurlyeq_i$ is equivalent to the quasi-concavity of utility function u_i. Also notice that the continuity of $\succcurlyeq_i$ is a sufficient condition for the continuous utility representations, that is, it guarantees the existence of continuous utility function $u_i(\cdot)$.

Remark 2.4.8 Under the convexity of preferences, $\succcurlyeq_i$, non-satiation implies local non-satiation. Why? The proof is lest to readers.

Now we are ready to answer under which conditions Walras's Law holds, a demand correspondence can be function, and convex-valued. The following propositions answer the questions.

Proposition 2.4.1 *Under local no-satiation assumption, we have the budget constraint holds with equality, and thus the Walras's Law holds.*

Proposition 2.4.2 *Under the strict convexity of* $\succcurlyeq_i$, $x_i(p)$ *becomes a (single-valued) function.*

Proposition 2.4.3 *Under the weak convexity of preferences, demand correspondence* $x_i(p)$ *is convex-valued.*

Strict convexity of production set: if $y_j^1 \in Y_j$ and $y_j^2 \in Y_j$, then the convex combination $\lambda y_j^1 + (1-\lambda)y_j^2 \in intY_j$ for all $0 < \lambda < 1$, where $intY_j$ denotes the interior points of Y_j.

The proof of the following proposition is based on the maximum theorem.

Proposition 2.4.4 *If* Y_j *is compact (i.e., closed and bounded) and strictly convex, then the supply correspondence* $y_j(p)$ *is a well defined single-valued and continuous function.*

Proof: By the maximum theorem, we know that $y_j(p)$ is a non-empty valued upper hemi-continuous correspondence by the compactness of y_j (by noting that $0 \in Y_j$) for all $p \in R^L_+$.

Now we show it is single-valued. Suppose not. y_j^1 and y_j^2 are two profit maximizing production plans for $p \in \Re^L_+$, and thus $py_j^1 = py_j^2$. Then, by the strict convexity of Y_j, we have $\lambda y_j^1 + (1-\lambda)y_j^2 \in \text{int}Y_j$ for all $0 < \lambda < 1$. Therefore, there exists some $t > 1$ such that

$$t[\lambda y_j^1 + (1-\lambda)y_j^2] \in \text{int}Y_j. \tag{2.24}$$

Then $t[\lambda p y_j^1 + (1-\lambda)p y_j^2] = tpy_j^1 > py_j^1$ which contradicts the fact that y_j^1 is a profit maximizing production plan.

So $y_j(p)$ is a single-valued function. Thus, by the upper hemi-continuity of $y_j(p)$, we know it is a single-valued and continuous function.

Proposition 2.4.5 *If $\succcurlyeq_i$ is continuous, strictly convex, locally non-satiated, and $w_i > 0$, then $x_i(p)$ is a continuous single-valued function and satisfies the budget constraint with equality. As such the Walras's Law is satisfied.*

Proof. First note that, since $w_i > 0$, one can show that the budget constrained set $B_i(p)$ is a continuous correspondence with non-empty and compact values and $\succcurlyeq_i$ is continuous. Then, by the maximum theorem, we know the demand correspondence $x_i(p)$ is upper hemi-continuous. Furthermore, by the strict convexity of preferences, it is single-valued and continuous. Finally, by local non-satiation, we know the budget constraint holds with equality, and thus Walras's Law is satisfied.

Note that there's no utility function representation when preferences are lexicographic.

Then, from the above propositions, we can have the following existence theorem that provides sufficient conditions directly based on the fundamentals of the economy by applying the Existence Theorem I above.

Theorem 2.4.7 *For a private ownership economy $e = (\{X_i, w_i, \succcurlyeq_i\}, \{Y_j\}, \{\theta_{ij};\})$, there exists a competitive equilibrium if the following conditions hold*

(i) $X_i \in \Re_+^L$;

(ii) $\succcurlyeq_i$ *are continuous, strictly convex (which guarantee the demand function is single valued) and locally non-satiated (which guarantees Walras' Law holds);*

(iii) $w_i > 0$;

(iv) Y_j *are compact, strictly convex,* $0 \in Y_j$ $j = 1, 2, \ldots, J$.

Note that $\succcurlyeq_i$ are continuous if the upper contour set $U_w(x_i) \equiv \{x_i' \in X_i \text{ and } x_i' \succcurlyeq_i x_i\}$ and the lower contour set $L_w(x_i) \equiv \{x_i' \in X_i \text{ and } x_i' \preccurlyeq_i x_i\}$ are closed.

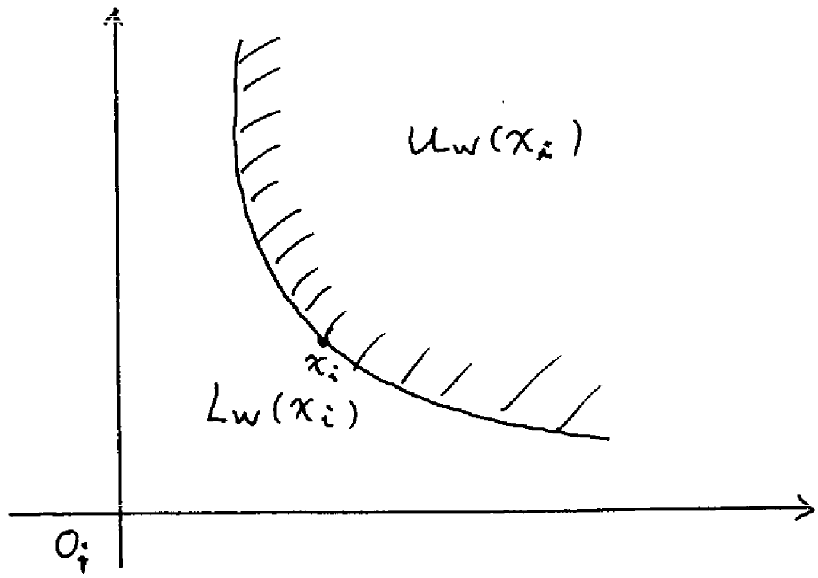

Figure 2.14: The upper contour set $U_w(x_i)$ is given by all points above the indifference curve, and the lower contour set $L_w(x_i)$ is given by all points below the indifference curve in the figure.

Proof: By the assumption imposed, we know that $x_i(p)$ and $y_j(p)$ are continuous single-valued. Thus the aggregate excess demand function is a continuous function and satisfies Walras' Law, and therefore, there is a competitive equilibrium.

The above results require the aggregate excess demand function is continuous. By using the KKM lemma, we can prove the existence of competitive equilibrium by only assuming the aggregate excess demand function is lower hemi-continuous.

Theorem 2.4.8 (The Existence Theorem I′) *For a private ownership economy $e = (\{X_i, w_i, \succcurlyeq_i\}, \{Y_j\}, \{\theta_{ij}\})$, if the aggregate excess demand function $\hat{z}(p)$ is a lower semi-continuous correspondence and satisfies Walras' Law, then there exists a competitive equilibrium, that is, there is $p^* \in S^{L-1}$ such that*

$$\hat{z}(p)^* \leqq 0. \tag{2.25}$$

Proof. Define a correspondence $F : S \to 2^S$ by,

$$F(q) = \{p \in S : q\hat{z}(p) \leqq 0\} \text{ for all } q \in S.$$

Since $p \geqq 0$ and $\hat{z}(\cdot)$ is lower semi-continuous, the function defined by $\phi(q,p) \equiv q\hat{z}(p) = \sum_{l=1}^{L} q^l \hat{z}^l(p)$ is lower semi-continuous in p. Hence, the set $F(q)$ is closed for all $q \in S$. We now prove F is FS-convex. Suppose, by way of contradiction, that there some $q_1, \ldots, q_m \in S$ and some convex combination $q_\lambda = \sum_{t=1}^{m} \lambda_t q_t$ such that $q_\lambda \notin \cup_{t=1}^{m} F(q_t)$. Then, $q_\lambda \notin F(q_t)$ for all $t = 1, \ldots, m$. Thus, $\sum_{t=1}^{m} \lambda_t q_t \hat{z}(q_\lambda) = q_\lambda \hat{z}(q_\lambda) > 0$ which contradicts the fact that $\hat{z}$ satisfies Walras' Law. So F must be FS-convex. Therefore, by KKM lemma, we have

$$\cap_{q \in S} F(q) \neq \emptyset.$$

Then there exists a $p^* \in S$ such that $p^* \in \cap_{q \in S} F(q)$, i.e., $p^* \in F(q)$ for all $q \in S$. Thus, $q\hat{z}(p^*) \leqq 0$ for all $q \in S$. Now let $q_1 = (1, 0, \ldots, 0)$, $q_2 = (0, 1, 0, \ldots, 0)$, and $q_n = (0, \ldots, 0, 1)$. Then $q_t \in S$ and thus $q_t \hat{z}(p^*) = \hat{z}^t(p^*) \leqq 0$ for all $t = 1, \ldots, L$. Thus we have $\hat{z}(p^*) \leqq 0$, which means p^* is a competitive equilibrium price vector. The proof is completed.

Examples of Computing CE

Example 2.4.2 Consider an exchange economy with two consumers and two goods with

$$\begin{aligned} &w_1 = (1, 0) && w_2 = (0, 1) \\ &u_1(x_1) = (x_1^1)^a (x_1^2)^{1-a} && 0 < a < 1 \\ &u_2(x_2) = (x_2^1)^b (x_2^2)^{1-b} && 0 < b < 1 \end{aligned} \tag{2.26}$$

Let $p = \frac{p^2}{p^1}$

Consumer 1's problem is to solve

$$\max_{x_1} u_1(x_1) \tag{2.27}$$

subject to

$$x^1 + px^2 = 1 \tag{2.28}$$

Since utility functions are Cobb-Douglas types of functions, the solution is then given by

$$x^1(p) = \frac{a}{1} = a \tag{2.29}$$

$$x_1^2(p) = \frac{1-a}{p}. \tag{2.30}$$

Consumer 2's problem is to solve

$$\max_{x_2} u_2(x_2) \tag{2.31}$$

subject to

$$x_2^1 + px_2^2 = p. \tag{2.32}$$

The solution is given by

$$x_2^1(p) = \frac{b \cdot p}{1} = b \cdot p \tag{2.33}$$

$$x_2^2(p) = \frac{(1-b)p}{p} = (1-b). \tag{2.34}$$

Then, by the market clearing condition,

$$x_1^1(p) + x_2^1(p) = 1 \Rightarrow a + bp = 1 \tag{2.35}$$

and thus the competitive equilibrium is given by

$$p = \frac{p^2}{p^1} = \frac{1-a}{b}. \tag{2.36}$$

This is true because, by Walras's Law, for $L = 2$, it is enough to show only one market clearing.

Remark 2.4.9 Since the Cobb-Douglas utility function is widely used as an example of utility functions that have nice properties such as strict monotonicity on $\mathbb{R}^L_{++}$, continuity, and strict quasi-concavity, it is very useful to remember the functional form of the demand function derived from the Cobb-Douglas utility functions. It may be remarked that we can easily derive the demand function for the general function: $u_i(x_2) = (x_1^1)^\alpha (x_1^2)^\beta \quad \alpha > 0, \beta > 0$ by the a suitable monotonic transformation. Indeed, by invariant to monotonic transformation of utility function, we can rewrite the utility function as

$$[(x_1^1)^\alpha (x_1^2)^\beta]^{\frac{1}{\alpha+\beta}} = (x_1^1)^{\frac{\alpha}{\alpha+p}} (x_1^2)^{\frac{\beta}{\alpha+\beta}} \tag{2.37}$$

so that we have

$$x_1^1(p) = \frac{\frac{\alpha}{\alpha+\beta} I}{p^1} \tag{2.38}$$

and

$$x_1^2(p) = \frac{\frac{\beta}{\alpha+\beta} I}{p^2} \tag{2.39}$$

when the budget line is given by

$$p^1 x_1^1 + p^2 x_1^2 = I. \tag{2.40}$$

Example 2.4.3

$$\begin{aligned} &n = 2 && L = 2 \\ &w_1 = (1,0) && w_2 = (0,1) \\ &u_1(x_1) = (x_1^1)^a (x_1^2)^{1-a} && 0 < a < 1 \\ &u_2(x_2) = \min\{x_2^1, bx_2^2\} && \text{with } b > 0 \end{aligned} \tag{2.41}$$

For consumer 1, we have already obtained

$$x_1^1(p) = a, \qquad x_1^2 = \frac{(1-a)}{p}. \tag{2.42}$$

For Consumer 2, his problem is to solve:

$$\max_{x_2} u_2(x_2) \tag{2.43}$$

s.t.

$$x_2^1 + px_2^2 = p. \tag{2.44}$$

At the optimal consumption, we have

$$x_2^1 = bx_2^2. \tag{2.45}$$

By substituting the solution into the budget equation, we have $bx_2^2 + px_2^2 = p$ and thus $x_2^2(p) = \frac{p}{b+p}$ and $x_2^1(p) = \frac{bp}{b+p}$.

Then, by $x_1^1(p) + x_2^1(p) = 1$, we have

$$a + \frac{bp}{b+p} = 1 \tag{2.46}$$

or

$$(1-a)(b+p) = bp \tag{2.47}$$

so that

$$(a+b-1)p = b(1-a) \tag{2.48}$$

Thus,

$$p^* = \frac{b(1-a)}{a+b-1}.$$

To make p^* be a competitive equilibrium price, we need to assume $a + b > 1$.

2.4.3 The Existence of CE for Aggregate Excess Demand Correspondences

When preferences and/or production sets are not strictly convex, the demand correspondence and/or supply correspondence may not be single-valued, and thus the aggregate excess demand correspondence may not be single-valued. As a result, one cannot use the above existence results to argue the existence of competitive equilibrium. Nevertheless, by using the KKM lemma, we can still prove the existence of competitive equilibrium when the aggregate excess demand correspondence satisfies certain conditions.

Theorem 2.4.9 (The Existence Theorem II)) *For a private ownership economy $e = (\{X_i, w_i, \succcurlyeq_i\}, \{Y_j\}, \{\theta_{ij}\})$, if $\hat{z}(p)$ is an non-empty convex and compact-valued upper hemi-continuous correspondence and satisfies Walras' Law, then there exists a competitive equilibrium, that is, there is a price vector $r^* \in S$ such that*

$$\hat{z}(p)^* \cap \{-\Re^L\} \neq \emptyset. \tag{2.49}$$

Proof. Define a correspondence $F : S \to 2^S$ by,

$$F(q) = \{p \in S : q\hat{z} \leqq 0 \text{ for some } \hat{z} \in \hat{z}(p)\}.$$

Since $\hat{z}(\cdot)$ is upper semi-continuous, $F(q)$ is closed for each $q \in S$. We now prove F is FS-convex. Suppose, by way of contradiction, that there are some $q_1, \ldots, q_m \in S$ and some convex combination $q_\lambda = \sum_{t=1}^m \lambda_t q_t$ such that $q_\lambda \notin \cup_{t=1}^m F(q_t)$. Then, $q_\lambda \notin F(q_t)$ for all $t = 1, \ldots, m$. Thus, for all $\hat{z} \in \hat{z}(p)$, we have $q_\lambda \hat{z} > 0$ for $t = 1, \ldots, m$. Thus, $\sum_{t=1}^m \lambda_t q_t \hat{z} = q_\lambda \hat{z} > 0$ which contradicts the fact that $\hat{z}$ satisfies Walras' Law. So F must be FS-convex. Therefore, by KKM lemma, we have

$$\cap_{q \in S} F(q) \neq \emptyset.$$

Then there exists a $p^* \in S$ such that $p^* \in \cap_{q \in S} F(q)$, i.e., $p^* \in F(q)$ for all $q \in S$. Thus, for each $q \in S$, there is $\hat{z} \in \hat{z}(p^*)$ such that

$$q\hat{z} \leqq 0.$$

We now prove $\hat{z}(p^*) \cap \{-\Re^L_+\} \neq 0$. Suppose not. Since $\hat{z}(p^*)$ is convex and compact and $-\Re^L_+$ is convex and closed, by the Separating Hyperplane theorem, there exists some

$c \in \Re^L$ such that

$$q \cdot (-\Re^L_+) < c < q\hat{z}(p^*)$$

Since $(-\Re^L_+)$ is a cone, we must have $c > 0$ and $q \cdot (-\Re^L_+) \leqq 0$. Thus, $q \in \Re^L_+$ and $q\hat{z}(p^*) > 0$ for all q, a contradiction. The proof is completed.

Remark 2.4.10 The last part of the proof can be also shown by applying the following result: Let K be a compact convex set. Then $K \cap \{-\Re^L_+\} \neq \emptyset$ if and only if for any $p \in \Re^L_+$, there exists $z \in K$ such that $pz \leqq 0$. The proof of this result can be found, for example, in the book of K. Border (1985, p. 13).

Similarly, we have the following existence theorem that provides sufficient conditions directly based on economic environments by applying the Existence Theorem II above.

Theorem 2.4.10 *For a private ownership economy $e = (\{X_i, w_i, \succcurlyeq_i\}), \{Y_j\}, \{\theta_{ij}\}$, there exists a competitive equilibrium if the following conditions hold*

(i) $X_i \in \Re^L_+$;

(ii) $\succcurlyeq_i$ are continuous, weakly convex and locally non-satiated;

(iii) $w_i > 0$;

(iv) Y_j are compact, convex, and $0 \in Y$; $j = 1, 2, \ldots, J$.

2.4.4 The Existence of CE for General Production Economies

For a general private ownership production economy,

$$e = (\{X_i, \succcurlyeq_i, w_i\}, \{Y_j\}, \{\theta_{ij}\}) \tag{2.50}$$

recall that a competitive equilibrium consists of a feasible allocation (x^*y^*) and a price vector $p^* \in R^L_+$ such that

(i) $x^*_i \in D_i(p^*) \equiv x_i(p^*)$ (utility maximization, $i = 1, 2, \ldots n$)

(ii) $y^*_i \in S_i(p^*) \equiv y_i(p^*)$ (profit maximization, $j = 1, 2, \ldots J$)

We now state the following existence theorem for general production economies without proof. Since the proof is very complicated, we refer readers to the proof that can be found in Debreu (1959) who used the existence theorem on equilibrium of the abstract economy on Section 2.7.

Theorem 2.4.11 (Existence Theorem III, Debreu, 1959) *A competitive equilibrium for the private-ownership economy e exists if the following conditions are satisfied:*

(1) X_i is closed, convex and bounded from below;

(2) $\succcurlyeq_i$ are non-satiated;

(3) $\succcurlyeq_i$ are continuous;

(4) $\succcurlyeq_i$ are convex;

(5) $w_i \in intX_i$;

(6) $0 \in Y_j$ (possibility of inaction);

(7) Y_j are closed and convex (continuity and no IRS)

(8) $Y_j \cap \{-Y_j\} = \{0\}$ (Irreversibility)

(9) $\{-\Re^L_+\} \subseteq Y_j$ (free disposal)

2.5 The Uniqueness of Competitive Equilibria

We can easily give examples in which there are multiple competitive equilibrium price vectors. When is there only one normalized price vector that clears all markets?

The free goods case is not of great interest here, so we will rule it out by means of the desirability assumption so that every equilibrium price of each good must be strictly positive. We want to also assume the continuous differentiability of the aggregate excess demand function. The reason is fairly clear; if indifference curves have kinks in them, we can find whole ranges of prices that are market equilibria. Not only are the equilibria not unique, they are not even locally unique.

Thus, we answer this question for only considering the case of $p^* > 0$ and $\hat{z}(p)$ is differentiable.

Theorem 2.5.1 *If all goods are desirable and gross substitute for all prices (i.e., $\frac{\partial z^h(p)}{\partial p^l} > 0$ for $l \neq h$), then if p^* is a competitive equilibrium price vector and Walras' Law holds, it is the unique competitive equilibrium price vector.*

Proof: By the desirability, $p^* > 0$. Suppose p is another competitive equilibrium price vector that is not proportional to p^*. Let $m = \max \frac{p^l}{p^{*l}} = \frac{p^k}{p^{*k}}$ for some k. By homogeneity and the definition of competitive equilibrium, we know that $\hat{z}(p^*) = \hat{z}(mp^*) = 0$. We know that $m = \frac{p^k}{p^{*k}} \geqq \frac{p^l}{p^{*l}}$ for all $l = 1, \ldots, L$ and $m > \frac{p^h}{p^{*h}}$ for some h. Then we have $mp^{*l} \geqq p^l$ for all l and $mp^{*h} > p^h$ for some h. Thus, when the price of good k is fixed, the prices of the other goods are down. We must have the demand for good k down by the gross substitutes. Hence, we have $\hat{z}^k(p) < 0$, a contradiction.

2.6 Stability of Competitive Equilibrium

The concept of competitive equilibrium is a stationary concept. But, it has given no guarantee that the economy will actually operate at the equilibrium point. What forces exist that might tend to move prices to a market-clearing price? This is a topic about the stability on the price adjustment mechanism in a competitive equilibrium.

A paradoxical relationship between the idea of competition and price adjustment is that: If all agents take prices as given, how can prices move? Who is left to adjust prices?

To solve this paradox, one introduces a "Walrasian auctioneer" whose sole function is to seek for the market clearing prices. The Walrasian auctioneer is supposed to call the prices and change the price mechanically responding to the aggregate excess demand till the market clears. Such a process is called Tâtonnement adjustment process.

Tâtonnement Adjustment Process is defined, according to the laws of demand and supply, by

$$\frac{dp^l}{dt} = G^l(\hat{z}(p)) \quad l = 1, \ldots, L \tag{2.51}$$

where G^l is a sign-preserving function of $\hat{z}(p)$, i.e., $G^l(x) > 0$ if $x > 0$, $G^l(x) = 0$ if $x = 0$, and $G^l(x) < 0$ if $x < 0$. The above equation implies that when the aggregate excess demand is positive, we have a shortage and thus price should go up by the laws of demand and supply.

As a special case of G^l, G^l can be an identical mapping such that

$$\dot{p}^l = \hat{z}^l(p) \tag{2.52}$$

$$\dot{p} = \hat{z}(p). \tag{2.53}$$

Under Walras' Law,

$$\begin{aligned} \frac{d}{dt}(p'p) &= \frac{d}{dt}\left[\sum_{l=1}^{L}(p^l)^2\right] = 2\sum_{l=1}^{L}(p^l)\cdot\frac{dp^l}{dt} \\ &= 2p'\dot{p} = p\hat{z}(p) = 0 \end{aligned}$$

which means that the sum of squares of the prices remain constant as the price adjusts. This is another price normalization. The path of the prices are restricted on the surface of a k-dimensional sphere.

Examples of price dynamics in figures: The first and third figures show a stable equilibrium, the second and fourth figures show a unique unstable equilibrium.

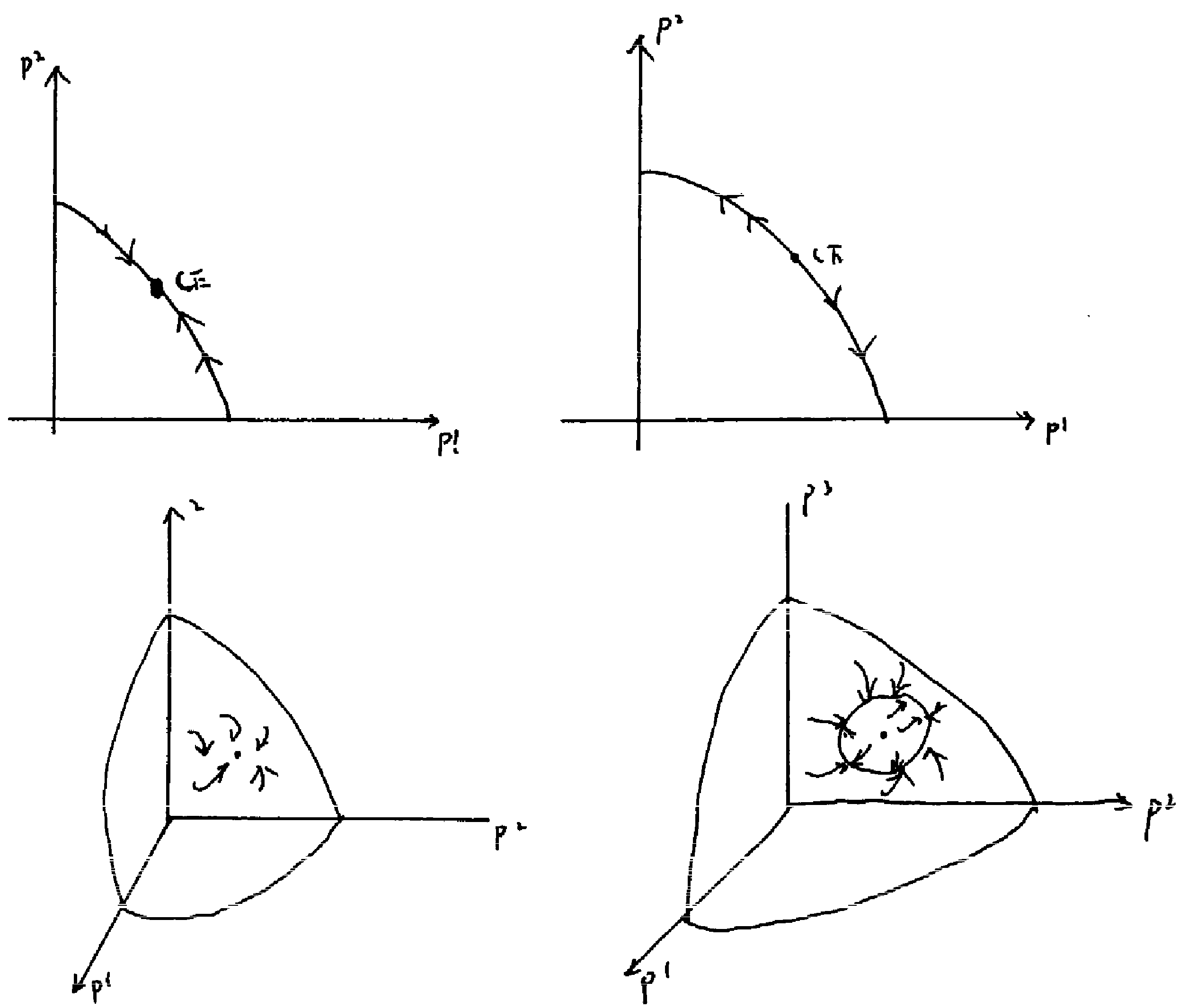

Figure 2.15: The first and third figures show the CEs are stable, and the second and fourth show they are not stable.

Definition 2.6.1 An equilibrium price p^* is globally stable if

(i) p^* is the unique competitive equilibrium,

(ii) for all p_o there exists a unique price path $p = \phi(t, p_0)$ for $0 \leqq t < \infty$ such that $\lim_{t\to\infty} \phi(t, p_o) = p^*$.

Definition 2.6.2 An equilibrium price p^* is locally stable if there is $\delta > 0$ and a unique price path $p = \phi(t, p_0)$ such that $\lim_{t\to\infty} \phi(t, p_o) = p^*$ whenever $|p - p_0| < \delta$.

The local stability of a competitive equilibrium can be easily obtained from the standard result on the local stability of a differentiate equation.

Theorem 2.6.1 *A competitive equilibrium price p^* is locally stable if the Jacobean matrix defined by*

$$A = \left[\frac{\partial \hat{z}^l(p^*)}{\partial p_k}\right]$$

has all negative characteristic roots.

The global stability result can be obtained by Liaponov Theorem.

Liaponov's function: For a differentiate equation system $\dot{x} = f(x)$ with $x^* = 0$, a function V is said to be a Liaponov's function if

(1) there is a unique x^* such that $V(x^*) = 0$;

(2) $V(x) > 0$ for all $x \neq x^*$;

(3) $\frac{dV(x)}{dt} < 0$.

Theorem 2.6.2 (Liaponov's Theorem) *If there exists a Liaponov's function for $\dot{x} = f(x)$, the unique stationary point x^* is globally stable.*

Debreu (1974) has shown essentially that any continuous function which satisfies the Walras' Law is an aggregate demand function for some economy. Thus, utility maximization places no restriction on aggregate behavior. Thus, to get global stability, one has to make some special assumptions.

The Weak Axiom of Revealed Preference (WARP) of the aggregate demand function: If $p\hat{z}(p) \geqq p\hat{z}(p')$, then $p'\hat{z}(p) > p'\hat{z}(p')$ for all $p, p' \in \Re^L_+$.

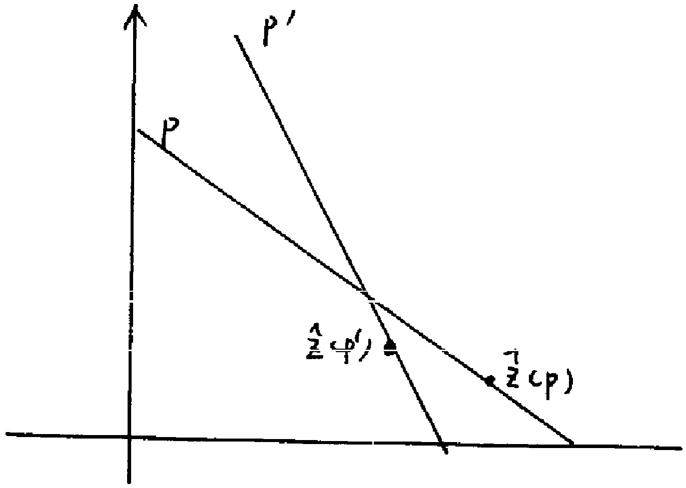

Figure 2.16: The figure shows an aggregate demand function satisfies WARP.

WARP implies that, if $\hat{z}(p')$ could have been bought at p where $\hat{z}(p)$ was bought, then at price p', $\hat{z}(p)$ is outside the budget constraint.

The WARP is a weak restriction than the continuous concave utility function. However, the restriction on the aggregate excess is not as weak as it may been seen. Even though two individuals satisfy the individual WARP, the aggregate demand function may not satisfy the aggregate WARP as shown in the following figures.

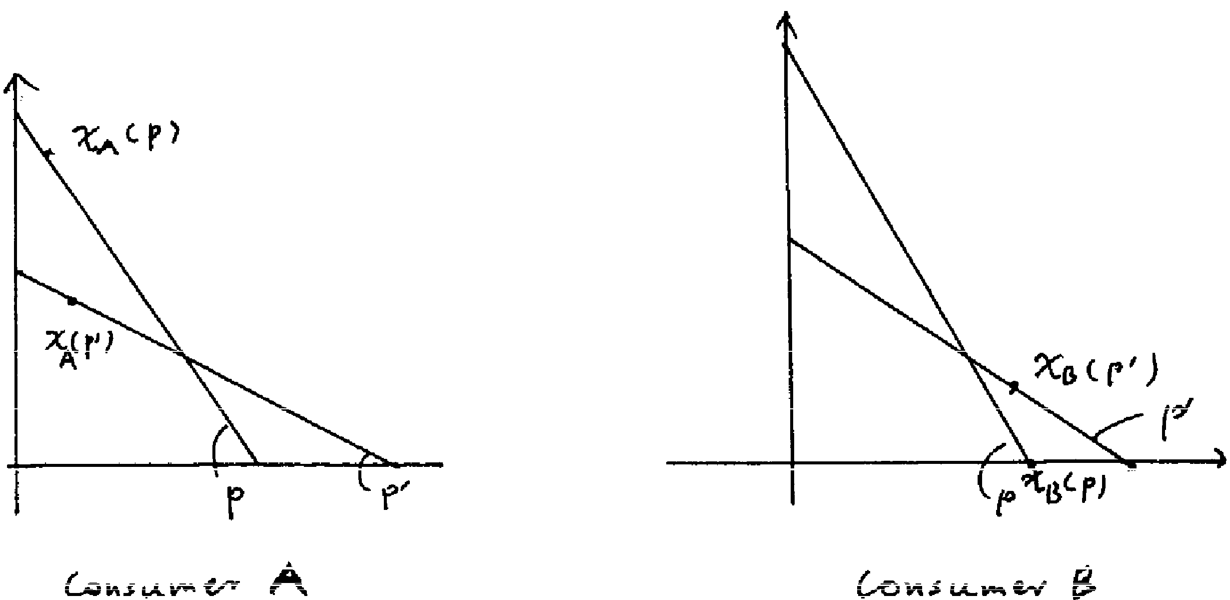

Figure 2.17: Both individual demand functions satisfy WARP.

Even if the individual demand functions satisfy WARP, it does not necessarily satisfy the WARP in aggregate.

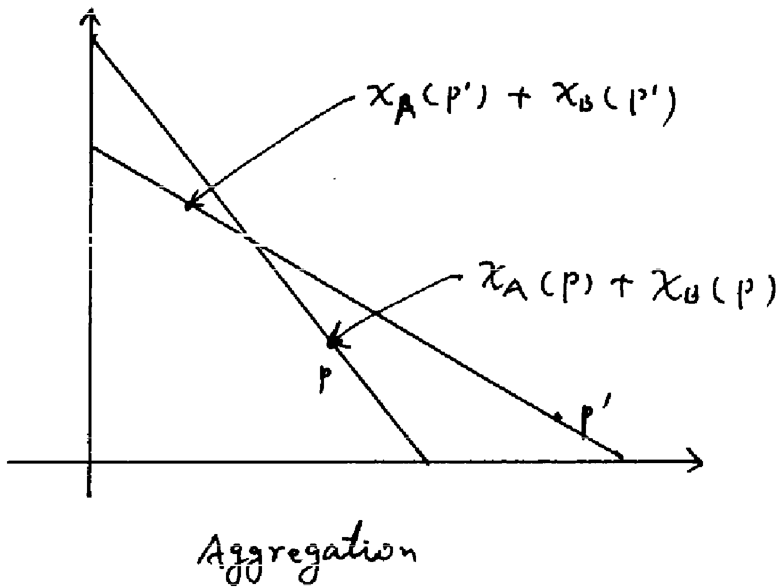

Figure 2.18: The aggregate excess demand function does not satisfy WARP.

Lemma 2.6.1 *Under the assumptions of Walras' Law and WARP, we have* $p^*\hat{z}(p) > 0$ *for all* $p \neq kp^*$ *where* p^* *is a competitive equilibrium.*

Proof: Suppose p^* is a competitive equilibrium.

$$\hat{z}(p^*) \leqq 0 \tag{2.54}$$

Also, by Walras' Law $p\hat{z}(p) = 0$. So we have $p\hat{z}(p) \geqq p\hat{z}(p^*)$. Then, by WARP, $p^*\hat{z}(p) > p^*\hat{z}(p^*) = 0 \Rightarrow p^*\hat{z}(p) > 0$ for all $p \neq kp^*$.

Lemma 2.6.2 *Under the assumptions of Walras' Law and WARP in aggregate, the competitive equilibrium is unique.*

Proof: By Lemma 2.6.1, for any $p \neq kp^*$, $p^*\hat{z}(p) > 0$ which means at least for some l, $\hat{z}^l > 0$.

Lemma 2.6.3 *Under the assumptions of Walras' Law and gross substitute, we have* $p^*\hat{z}(p) > 0$ *for all* $p \neq kp^*$ *where* $p^* > 0$ *is a competitive equilibrium.*

Proof. The proof of this lemma is complicated. We illustrate the proof only for the case of two commodities by aid of the figure. The general proof of this lemma can be seen in Arrow and Hahn, 1971, p. 224.

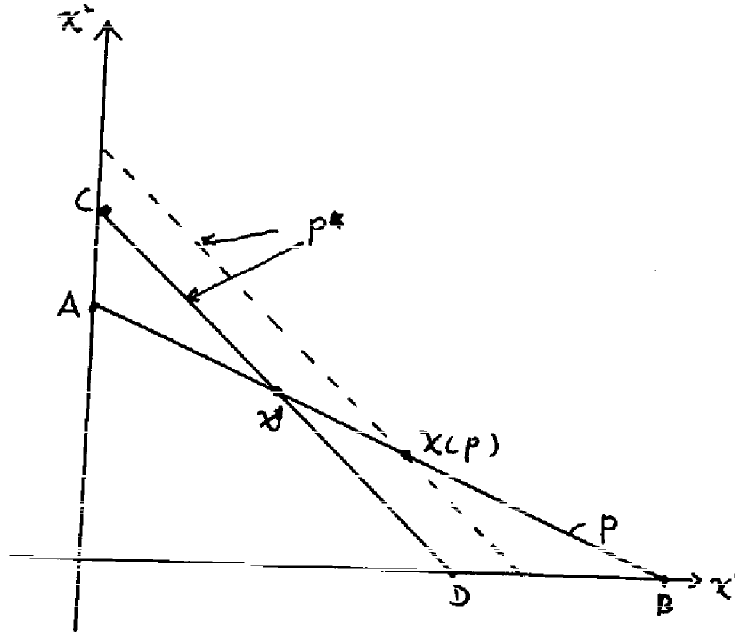

Figure 2.19: An illustration of the proof of Lemma 2.6.3.

Let p^* be an equilibrium price vector and $x^* = x(p^*)$. Let $p \neq \alpha p^*$ for any $\alpha > 0$. Then we know p is not an equilibrium price vector by the uniqueness of the competitive equilibrium under the gross substitutability. Since $p\hat{x}(p) = p\hat{x}(p^*)$ by Walras' Law, then $x(p)$ is on the line AB which passes through the point x^* and whose slope is given by p. Let CD be the line which passes through the point x^* and whose shope is p^*. We assume that $p^{*1}/p^{*2} > p^1/p^2$ without loss of generality. Then $p^{*1}/p^1 > p^{*2}/p^2 \equiv \mu$. Thus, $p^{*1} > \mu p^1$ and $p^{*2} = \mu p^2$. Therefore, we have $x^2(p^*) > x^2(\mu p)$ by the gross substitutability. But Walras' Law implies that $\mu p\hat{x}(\mu p) = \mu p\hat{x}(p^*)$ so that we must have $x^1(\mu p) > x^1(p^*)$. Using the homogeneity assumption, we get $x^1(p) = x^1(\mu p) > x^1(p^*) = x^{*1}$ and $x^2(p) = x^2(\mu p) < x^2(p^*) = x^{*2}$. Hence the point $x(p)$ must lie to the right of the point x^* in the figure. Now draw a line parallel to CD passing through the point $x(p)$. We see that $p^*\hat{x}(p) > p^*\hat{x}^*$ and thus $p^*\hat{z}(p) > 0$. The proof is completed.

Now we are ready to prove the following theorem on the global stability of a competitive equilibrium.

Theorem 2.6.3 (Arrow-Block-Hurwicz) *Under the assumptions of Walras' Law, if $\hat{z}(p)$ satisfies either gross substitutability, or WARP, then the competitive equilibrium price is globally stable.*

Proof: Define a Liaponov's function by

$$V(p) = \sum_{l=1}^{L} (p^l(t) - p^{*l})^2 = (p - p^*) \cdot (p - p^*) \tag{2.55}$$

By the assumption, the competitive equilibrium price p^* is unique. Also, since

$$\begin{aligned} \frac{dV}{dt} &= 2\sum_{l=1}^{L}(p(t)^l - p^{*l})\frac{dp^l_{(t)}}{dt} \\ &= 2\sum_{l=1}^{L}(p^l_{(t)} - p^{*l})\hat{z}^l_{(p)} \\ &= 2[p\hat{z}(p) - p^*\hat{z}(p)] \\ &= -2p^*\hat{z}(p) < 0 \end{aligned}$$

by Walras' Law and Lemma 2.6.1-2.6.3 for $p \neq kp^*$, we know $\dot{p} = \hat{z}(p)$ is globally stable by Leaponov's theorem.

The above theorem establishes the global stability of a competitive equilibrium under Walras' Law, homogeneity, and gross substitutability/WARP. It is natural to ask how far we can relax these assumptions, in particular gross substitutability. Since many goods in reality are complementary goods, can the global stability of a competitive equilibrium also hold for complementary goods? Scarf (1961) has constructed examples that show that a competitive equilibrium may not be globally stable.

Example 2.6.1 (Scarf's Counterexample on Global Instability) Consider a pure exchange economy with three consumers and three commodities (n=3, L=3). Suppose consumers' endowments are given by $w_1 = (1, 0, 0)$, $w_2 = (0, 1, 0)$ $w_3 = (0, 0, 1)$ and their utility functions are given by

$$\begin{aligned} u_1(x_1) &= \min\{x_1^1, x_1^2\} \\ u_2(x_2) &= \min\{x_2^2, x_2^3\} \\ u_3(x_3) &= \min\{x_3^1, x_3^3\} \end{aligned}$$

so that they have L-shaped indifference curves. Then, the aggregate excess demand function is given by

$$\hat{z}^1(p) = -\frac{p^2}{p^1+p^2} + \frac{p^3}{p^1+p^3} \tag{2.56}$$

$$\hat{z}^2(p) = -\frac{p^3}{p^2+p^3} + \frac{p^1}{p^1+p^2} \tag{2.57}$$

$$\hat{z}^3(p) = -\frac{p^1}{p^1+p^3} + \frac{p^2}{p^2+p^3}. \tag{2.58}$$

Then, the dynamic adjustment equation is given by

$$\dot{p} = \hat{z}(p).$$

We know that $\|p(t)\| =$ constant for all t by Walras' Law. Now we want to show that $\prod_{l=1}^{3} p^l(t) =$ constant for all t. Indeed,

$$\begin{aligned} \frac{d}{dt}(\prod_{l=1}^{3} p(t)) &= \dot{p}^1p^2p^3 + \dot{p}^2p^1p^3 + \dot{p}^3p^1p^2 \\ &= \hat{z}^1p^2p^3 + \hat{z}^2p^1p^3 + \hat{z}^3p^1p^2 = 0. \end{aligned}$$

Now, we show that the dynamic process is not globally stable. First choose the initial prices $p^l(0)$ such that $\sum_{l=1}^{3}[p^l(0)]^2 = 3$ and $\prod_{l=1}^{3} p^l(0) \neq 1$. Then, $\sum_{l=1}^{3}[p^l(t)]^2 = 3$ and $\prod_{l=1}^{3} p^l(t) \neq 1$ for all t. Since $\sum_{l=1}^{3}[p^l(t)]^2 = 3$, the only possible equilibrium prices are $p^{*1} = p^{*2} = p^{*3} = 1$, the solution of the above system of differential equations cannot converge to the equilibrium price $p^* = (1, 1, 1)$.

In this example, we may note the following facts. (i) there is no substitution effect, (ii) the indifference curve is not strictly convex, and (iii) the difference curve has a kink and hence is not differentiable. Scarf (1961) also provided the examples of instabiity in which the substitution effect is present for the case of Giffen's goods. Thus, Scarf's examples indicate that instability may occur in a wide variety of cases.

2.7 Abstract Economy

The abstract economy defined by Debreu (Econometrica, 1952) generalizes the notion of N-person Nash noncooperative game in that a player's strategy set depends on the strategy choices of all the other players and can be used to prove the existence of competitive equilibrium since the market mechanism can be regarded as an abstract economy as shown by Arrow and Debreu (Econometrica 1954). Debreu (1952) proved the existence of equilibrium in abstract economies with finitely many agents and finite dimensional strategy spaces by assuming the existence of continuous utility functions. Since Debreu's seminal work on abstract economies, many existence results have been given in the literature. Shafer and Sonnenschein (J. of Mathematical Economics, 1975) extended Debreu's results to abstract economies without ordered preferences.

2.7.1 Equilibrium in Abstract Economy

Let N be the set of agents which is any countable or uncountable set. Each agent i chooses a strategy x_i in a set X_i of $\mathbb{R}^L$. Denote by X the (Cartesian) product $\prod_{j\in N} X_j$ and X_{-i} the product $\prod_{j\in N\setminus\{i\}} X_j$. Denote by x and x_{-i} an element of X and X_{-i}. Each agent i has a payoff (utility) function $u_i: X \to \mathbb{R}$. Note that agent i's utility is not only dependent on his own choice, but also dependent on the choice of the others. Given x_{-i} (the strategies of others), the choice of the i-th agent is restricted to a non-empty, convex and compact set $F_i(x_{-i}) \subset X_i$, the *feasible strategy set*; the i-th agent chooses $x_i \in F_i(x_{-i})$ so as to maximize $u_i(x_{-i}, x_i)$ over $F_i(x_{-i})$.

An abstract economy (or called generalized game) $\Gamma = (X_i, F_i, u_i)_{i\in N}$ is defined as a family of ordered triples (X_i, F_i, P_i).

Definition 2.7.1 A vector $x^* \in X$ is said to be an *equilibrium of an abstract economy* if $\forall i \in N$

(i) $x_i^* \in F_i(x_{-i}^*)$ and

(ii) x_i^* maximizes $u_i(x_{-i}^*, x_i)$ over $F_i(x_{-i}^*)$.

If $F_i(x_{-i}) \equiv X_i$, $\forall i \in N$, the abstract economy reduces to the conventional game $\Gamma = (X_i, u_i)$ and the equilibrium is called a *Nash equilibrium.*

Theorem 2.7.1 (Arrow-Debreu) *Let X be a non-empty compact convex subset of $\mathbb{R}^{nL}$. Suppose that*

i) the correspondence $F\colon X \to 2^X$ is a continuous correspondence with non-empty compact and convex values,

ii) $u_i : X \times X \to \mathbb{R}$ is continuous, *iii)* $u_i : X \times X \to \mathbb{R}$ *is either quasi-concave in* x_i *or it has a unique maximum on* $F_i(x_{-i})$ *for all* $x_{-i} \in X_{-i}$.

Then Γ *has an equilibrium.*

Proof. For each $i \in N$, define the maximizing correspondence

$$M_i(x_{-i}) = \{x_i \in F_i(x_{-i}) : u_i(x_{-i}, x_i) \geqq u_i(x_{-i}, z_i),\ \forall z_i \in F_i(x_{-i})\}.$$

Then, the correspondence $M_i : X_{-i} \to 2^{X_i}$ is non-empty and convex-valued because u_i is continuous in x and quasi-concave in x_i and $F_i(x_{-i})$ is non-empty convex compact-valued. Also, by the Maximum Theorem, M_i is an upper hemi-continuous correspondence with compact-valued. Therefore the correspondence

$$M(x) = \prod_{i \in N} M_i(x_{-i})$$

is an upper hemi-continuous correspondence with non-empty convex compact-values. Thus, by Kakutani's Fixed Point Theorem, there exists $x^* \in X$ such that $x^* \in M(x^*)$ and x^* is an equilibrium in the generalized game. Q.E.D

A competitive market mechanism can be regarded as an abstract economy. For simplicity, consider an exchange economy $e = (X_i, u_i, w_i)_{i \in N}$. Define an abstract economy $\Gamma = (Z, F_i, u_i)_{i \in N+1}$ as follows. Let

$$\begin{aligned}
Z_i &= X_i \quad i = 1, \ldots, n \\
Z_{n+1} &= \Delta^{L-1} \\
F_i(x_i, p) &= \{x_i \in X_i : px_i \leqq pw_i\} \quad i = 1, \ldots, n \\
F_{n+1} &= \Delta^{L-1} \\
u_{n+1}(p, x) &= \sum_{i=1}^{n} p(x_i - w_i)
\end{aligned} \tag{2.59}$$

Then, we verify that the economy e has a competitive equilibrium if the abstract economy defined above has an equilibrium by noting that $\sum_{i=1}^n x_i \leqq \sum_{i=1}^n w_i$ at the equilibrium of the abstract equilibrium.

2.7.2 The Existence of Equilibrium for General Preferences

The above theorem on the existence of an equilibrium in abstract economy has assumed the preference relation is an ordering and can be represented by a utility function. In this subsection, we consider the existence of equilibrium in an abstract economy where individuals' preferences $\succcurlyeq$ may be non total or not-transitive. Define agent i's preference correspondence $P_i : X \to 2^{X_i}$ by

$$P_i(x) = \{y_i \in X_i : (y_i, x_{-i}) \succ_i (x_i, x_{-i})\}$$

We call $\Gamma = (X_i, F_i, P_i)_{i \in N}$ an abstract economy.

A *generalized game (or an abstract economy)* $\Gamma = (X_i, F_i, P_i)_{i \in N}$ is defined as a family of ordered triples (X_i, F_i, P_i). An *equilibrium* for Γ is an $x^* \in X$ such that $x^* \in F(x^*)$ and $P_i(x^*) \cap F_i(x^*) = \emptyset$ for each $i \in N$.

Shafer and Sonnenschein (1975) proved the following theorem that generalizes the above theorem to an abstract economy with non-complete/non-transitive preferences.

Theorem 2.7.2 (Shafer-Sonnenschein) *Let $\Gamma = (X_i, F_i, P_i)_{i \in N}$ be an abstract economy satisfying the following conditions for each $i \in N$:*

(i) *X_i is a non-empty, compact, and convex subset in $\mathbb{R}^{l_i}$,*

(ii) *$F_i : X \to 2^{X_i}$ is a continuous correspondence with non-empty, compact, and convex values,*

(iii) *P_i has open graph,*

(iv) *$x_i \notin con\ P_i(x)$ for all $x \in Z$.*

Then Γ has an equilibrium.

This theorem requires the preferences have open graph. Tian (International Journal of Game Theory, 1992) proved the following theorem that is more general and generalizes the results of Debreu (1952), Shafer and Sonnenschein (1975) by relaxing the openness of graphs or lower sections of preference correspondences. Before proceeding to the theorem, we state some technical lemmas which were due to Micheal (1956, Propositions 2.5, 2.6 and Theorem 3.1$'''$).

Lemma 2.7.1 *Let $X \subset \mathbb{R}^M$ and $Y \subset \mathbb{R}^K$ be two convex subsets and $\phi : X \to 2^Y$, $\psi : X \to 2^Y$ be correspondences such that*

(i) *ϕ is lower hemi-continuous, convex valued, and has open upper sections,*

(ii) *ψ is lower hemi-continuous,*

(iii) *for all $x \in X$, $\phi(x) \cap \psi(x) \neq \emptyset$.*

Then the correspondence $\theta : X \to 2^Y$ defined by $\theta(x) = \phi(x) \cap \psi(x)$ is lower hemi-continuous.

Lemma 2.7.2 *Let $X \subset \mathbb{R}^M$ and $Y \subset \mathbb{R}^K$ be two convex subsets, and let $\phi : X \to 2^Y$ be lower hemi-continuous. Then the correspondence $\psi : X \to 2^Y$ defined by $\psi(x) = con\,\phi(x)$ is lower hemi-continuous.*

Lemma 2.7.3 *Let $X \subset \mathbb{R}^M$ and $Y \subset \mathbb{R}^K$ be two convex subsets. Suppose $F : X \to 2^Y$ is a lower hemi-continuous correspondence with non-empty and convex values. Then there exists a continuous function $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.*

Theorem 2.7.3 (Tian) *Let $\Gamma = (X_i, F_i, P_i)_{i \in N}$ be a generalized game satisfying for each $i \in N$:*

(i) *X_i is a non-empty, compact, and convex subset in $\mathbb{R}^{l_i}$,*

(ii) *F_i is a continuous correspondence, and $F_i(x)$ is non-empty, compact, and convex for all $x \in X$,*

(iii) *P_i is lower hemi-continuous and has open upper sections,*

(iv) *$x_i \notin con\ P_i(x)$ for all $x \in F$.*

Then Γ has an equilibrium.

Proof. For each $i \in N$, define a correspondence $A_i : X \to 2^{X_i}$ by $A_i(x) = F_i(x) \cap con\, P_i(x)$. Let $U_i = \{x \in X : A_i(x) \neq \emptyset\}$. Since F_i and P_i are lower hemi-continuous in X, so are they in U_i. Then, by Lemma 2.7.2, $con\, P_i$ is lower hemi-continuous in U_i. Also since P_i has open upper sections in X, so does $con P_i$ in X and thus $con\, P_i$ in U_i. Further, $F_i(x) \cap con\, P_i(x) \neq \emptyset$ for all $x \in U_i$. Hence, by Lemma 2.7.1, the correspondence $A_i|U_i : U_i \to 2^{X_i}$ is lower hemi-continuous in U_i and for all $x \in U_i$, $F(x)$ is non-empty and convex. Also X_i is finite dimensional. Hence, by Lemma 2.7.3, there exists a continuous function

$f_i : U_i \to X_i$ such that $f_i(x) \in A_i(x)$ for all $x \in U_i$. Note that U_i is open since A_i is lower hemi-continuous. Define a correspondence $G_i : X \to 2^{X_i}$ by

$$G_i(x) = \begin{cases} \{f_i(x)\} & \text{if } x \in U_i \\ F_i(x) & \text{otherwise} \end{cases}. \tag{2.60}$$

Then G_i is upper hemi-continuous. Thus the correspondence $G : X \to 2^X$ defined by $G(x) = \prod_{i \in N} G_i(x)$ is upper hemi-continuous and for all $x \in X$, $G(x)$ is non-empty, closed, and convex. Hence, by Kakutani's Fixed Point Theorem, there exists a point $x^* \in X$ such that $x^* \in G(x^*)$. Note that for each $i \in N$, if $x^* \in U_i$, then $x_i^* = f_i(x^*) \in A(x^*) \subset con\, P_i(x^*)$, a contradiction to (iv). Hence, $x^* \notin U_i$ and thus for all $i \in N$, $x_i^* \in F_i(x^*)$ and $F_i(x^*) \cap con\, P_i(x^*) = \emptyset$ which implies $F_i(x^*) \cap P_i(x^*) = \emptyset$. Thus Γ has an equilibrium. ■

Note that a correspondence P has open graph implies that it has upper and lower open sections; a correspondence P has lower open sections implies P is lower hemi-continuous. Thus, the above theorem is indeed weaker.

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Chapter 3

Normative Theory of Equilibrium: Its Welfare Properties

3.1 Introduction

In the preceding chapter, we studied the conditions which would guarantee the existence, uniqueness, and stability of competitive equilibrium. In this chapter, we will study the welfare properties of competitive equilibrium.

Economists are interested not only in describing the world of the market economy, but also in evaluating it. Does a competitive market do a "good" job in allocating resources? Adam Smith's "invisible hand" says that market economy is efficient. Then in what sense and under what conditions is a market efficient? The concept of efficiency is related to a concern with the well-being of those in the economy. The normative analysis can not only help us understand what advantage the market mechanism has, it can also help us to evaluate an economic system used in the real world, as well as help us understand why China and East European countries want to change their economic institutions.

The term of economic efficiency consists of three requirements:

(1) Exchange efficiency: goods are traded efficiently so that no further mutual beneficial trade can be obtained.

(2) Production efficiency: there is no waste of resources in producing goods.

(3) Product mix efficiency, i.e., the mix of goods produced by the economy

reflects the preferences of those in the economy.

3.2 Pareto Efficiency of Allocation

When economists talk about the efficiency of allocation resources, it means Pareto efficiency. It provides a minimum criterion for efficiency of using resources.

Under the market institution, we want to know what is the relationship between a market equilibrium and Pareto efficiency. There are two basic questions: (1) If a market (not necessarily competitive) equilibrium exists, is it Pareto efficient? (2) Can any Pareto efficient allocation be obtained through the market mechanism by redistributing endowments?

The concept of Pareto efficiency is not just to study the efficiency of a market economy, but it also can be used to study the efficiency of any economic system.

Definition 3.2.1 (Pareto Efficiency, also called Pareto Optimality) *:* An allocation is Pareto efficient or Pareto optimal (in short, P.O) if it is feasible and there is no other feasible allocation such that one person would be better off and all other persons are not worse off.

More precisely, for exchange economies, a feasible allocation x is $P.O.$ if there is no other allocation x' such that

(i) $\sum_{i=1}^{n} x_i' \leqq \sum_{i=1}^{n} w_i$

(ii) $x_i' \succcurlyeq_i x_i$ for all i and $x_k' \succ_k x_k$ for some $k = 1, \ldots, n$.

For production economy, (x, y) is Pareto optimal if and only if:

(1) $\hat{x} \leqq \hat{y} + \hat{w}$

(2) there is no feasible allocation (x', y') s.t.

$$\begin{aligned} x_i' &\succcurlyeq x_i \text{ for all } i \\ x_k' &\succ_k x_k \text{ for some } k \end{aligned}$$

A weaker concept about economic efficiency is the so-called weak Pareto efficiency.

Definition 3.2.2 (Weak Pareto Efficiency) An allocation is *weakly Pareto efficient* if it is feasible and there is no other feasible allocation such that all persons are better off.

Remark 3.2.1 *Some textbooks such as Varian (1992) has used weak Pareto optimality as the definition of Pareto optimality. Under which conditions are they equivalent? It is clear that Pareto efficiency implies weak Pareto efficiency. But the converse may not be true. However, under the continuity and strict monotonicity of preferences, the converse is true.*

Proposition 3.2.1 *Under the continuity and strict monotonicity of preferences, weak Pareto efficiency implies Pareto efficiency.*

Proof. Suppose x is weakly Pareto efficient but not Pareto efficient. Then, there exists a feasible allocation x' such that $x'_i \succcurlyeq_i x_i$ for all i and $x'_k \succ_k x_k$ for some k.

Define $\overline{x}$ as follows

$$\begin{aligned}\overline{x}_k &= (1-\theta)x'_k \\ \overline{x}_i &= x'_i + \frac{\theta}{n-1}x'_k \text{ for } i \neq k\end{aligned}$$

Then we have

$$\overline{x}_k + \sum_{i\neq k}\overline{x}_i = (1-\theta)x'_k + \sum_{i\neq k}(x'_i + \frac{\theta}{n-1}x'_k) = \sum_{i=1}^{n} x'_i \tag{3.1}$$

which means $\overline{x}$ is feasible. Furthermore, by the continuity of preferences, $\overline{x}_k = (1-\theta)x'_k \succ x_k$ when θ is sufficiently close to zero, and $\bar{x}_i \succ_i x_i$ for all $i \neq k$ by the strict monotonicity of preferences. This contradicts the fact that x is weakly Pareto optimal. Thus, we must have every weak Pareto efficient allocation is Pareto efficient under the monotonicity and continuity of preferences. ■

Remark 3.2.2 The trick of taking a little from one person and then equally distribute it to the others will make every one better off is a useful one. We will use the same trick to prove the second theorem of welfare economics.

Remark 3.2.3 The above proposition also depends on an implicit assumption that the goods under consideration are all private goods. Tian (Economics Letters, 1988) showed, by example that, if goods are public goods, we may not have such equivalence between Pareto efficiency and weak Pareto efficiency under the monotonicity and continuity of preferences.

The set of Pareto efficient allocations can be shown with the Edgeworth Box. Every point in the Edgeworth Box is attainable. A is the starting point. Is "A" a weak Pareto optimal? No. The point C, for example in the shaded area is better off to both persons. Is the point "B" a Pareto optimal? YES. Actually, all tangent points are Pareto efficient points where there are no incentives for any agent to trade. The locus of all Pareto efficient is called the contract curve.

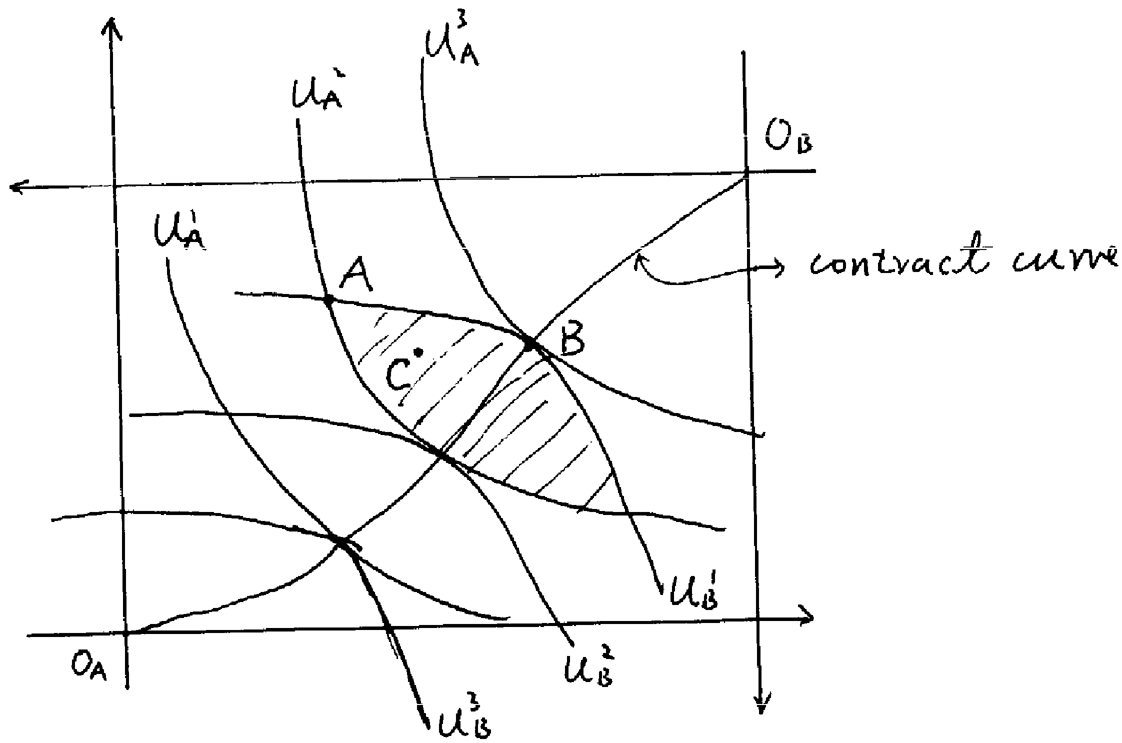

Figure 3.1: The set of Pareto efficient allocations is given by the contract curve.

Remark 3.2.4 *Equity* and *Pareto efficiency* are two different concepts. The points like O_A or O_B are Pareto optimal points, but these are extremely unequal. To make an allocation be relatively equitable and also efficient, government needs to implement some institutional arrangements such as tax, subsidy to balance between equity and efficiency, but this is a value judgement and policy issue.

We will show that, when Pareto efficient points are given by tangent points of two persons' indifference curves, it should satisfy the following conditions:

$$MRS^A_{x^1x^2} = MRS^B_{x^1x^2}$$

$$x_A + x_B = \hat{w}.$$

When indifference curves of two agents are never tangent, we have the following cases.

Case 1. For linear indifference curves with positive slope, indifference curves of agents may not be tangent. How can we find the set of Pareto efficient allocations in this case? We can do it by comparing the steepness of indifference curves.

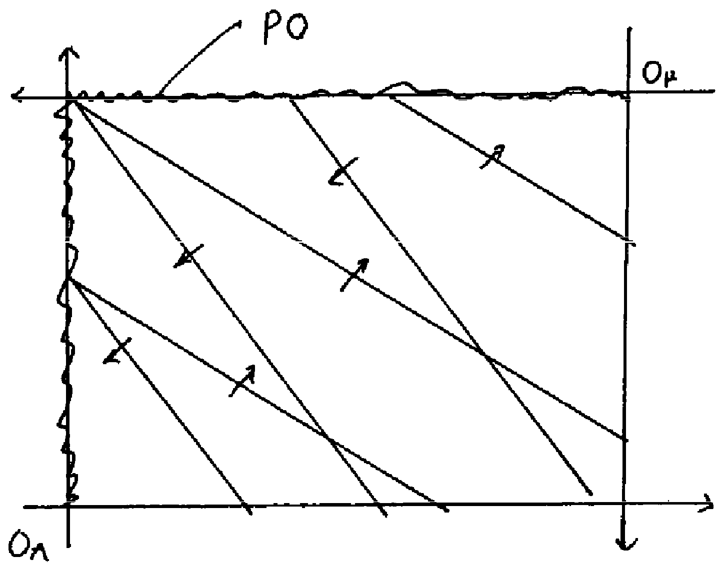

Figure 3.2: The set of Pareto efficient allocations is given by the upper and left edges of the box when indifference curves are linear.

$$MRS^A_{x^1x^2} < MRS^B_{x^1x^2} \tag{3.2}$$

In this case, when indifference curves for B is given, say, by Line AA, then K is a Pareto efficient point. When indifferent curves for A is given, say, by Line BB, then P is a Pareto efficient point. Contract curve is then given by the upper and left edge of the box.

Case 2. Suppose that indifference curves are given by

$$u_A(x_A) = x^2_A$$

and

$$u_B(x_B) = x^1_B$$

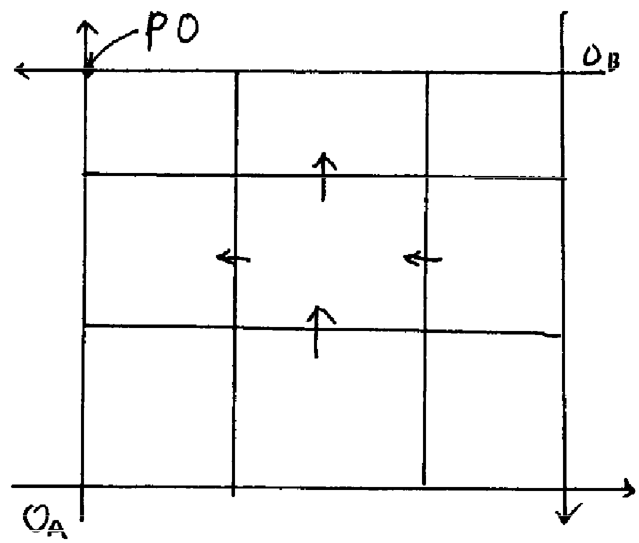

Figure 3.3: The only Pareto efficient allocation point is given by the upper and left corner of the box when individuals only care about one commodity.

Then, only Pareto efficient is the right upper corner point. But the set of weakly Pareto efficient is given by the upper and left edge of the box. Notice that utility functions in this example are continuous and monotonic, but a weak Pareto efficient allocation may not be Pareto efficient. This example shows that the strict monotonicity cannot be replaced by the monotonicity for the equivalence of Pareto efficiency and weak efficiency.

Case 3. Now suppose that indifference curves are perfect complementary. Then, utility functions are given by

$$u_A(x_A) = \min\{ax_A^1, bx_A^2\}$$

and

$$u_B(x_B) = \min\{cx_B^1, dx_B^2\}$$

A special case is the one where $a = b = c = d = 1$.

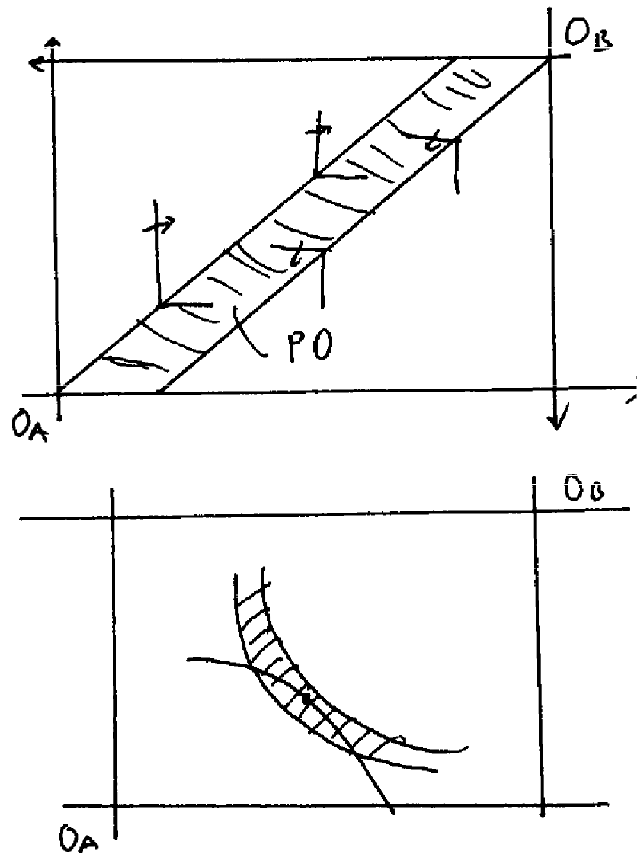

Figure 3.4: The first figure shows that the contract curve may be the "thick" when indifference curves are perfect complementary. The second figure shows that a weak Pareo efficient allocation may not Pareto efficient when indifference curves are "thick."

Then, the set of Pareto optimal allocations is the area given by points between two 45^0 lines.

Case 4. One person's indifference curves are "thick." In this case, an weak Pareto efficient allocation may not be Pareto efficient.

3.3 The First Fundamental Theorem of Welfare Economics

There is a well-known theorem, in fact, one of the most important theorems in economics, which characterizes a desirable nature of the competitive market institution. It claims that every competitive equilibrium allocation is Pareto efficient. A remarkable part of this theorem is that the theorem requires few assumptions, much fewer than those for the existence of competitive equilibrium. Some implicit assumptions in this section are that preferences are orderings, goods are divisible, and there are no public goods, or externalities.

Theorem 3.3.1 (The First Fundamental Theorem of Welfare Economics) *If (x, y, p) is a competitive equilibrium, then x is weakly Pareto efficient, and further under local non-satiation, it is Pareto efficient.*

Proof: Suppose (x, y) is not weakly Pareto optimal, then there exists another feasible allocation (x', y') such that $x'_i \succ_i x_i$ for all i. Thus, we must have $px'_i > px_i + \sum_{j=1}^J \theta_{ij} py_j$ for all i. Therefore, by summation, we have

$$\sum_{i=1}^n px'_i > \sum_{i=1}^n pw_i + \sum_{j=1}^J py_j.$$

Since $py_j \geqq py'_j$ for all $y'_j \in Y_j$ by profit maximization, we have

$$\sum_{i=1}^n px'_i > \sum_{i=1}^n pw_i + \sum_{j=1}^J py'_j. \tag{3.3}$$

or

$$p[\sum_{i=1}^n x'_i - \sum_{i=1}^n w_i - \sum_{j=1}^J y'_j] > 0, \tag{3.4}$$

which contradicts the fact that (x', y') is feasible.

To show Pareto efficiency, suppose (x, y) is not Pareto optimal. Then there exists another feasible allocation (x', y') such that $x'_i \succcurlyeq_i x_i$ for all i and $x'_k \succ_k x_k$ for some k . Thus we have, by local non-saltation,

$$px'_i \geqq pw_i + \sum_{j=1}^J \theta_{ij} py_j \quad \forall i$$

and by $x'_k \succ_k x_k$,

$$px'_k > pw_k + \sum_{j=1}^J \theta_{kj} py_j$$

and thus

$$\sum_{i=1}^n px'_i > \sum_{i=1}^n pw_i + \sum_{j=1}^J py_j \geqq \sum_{i=1}^n pw_i + \sum_{j=1}^J py'_j. \tag{3.5}$$

Again, it contradicts the fact that (x', y') is feasible. ■

Remark 3.3.1 If the local non-satiation condition is not satisfied, a competitive equilibrium allocation x may not be Pareto optimal, say, for the case of thick indifference curves.

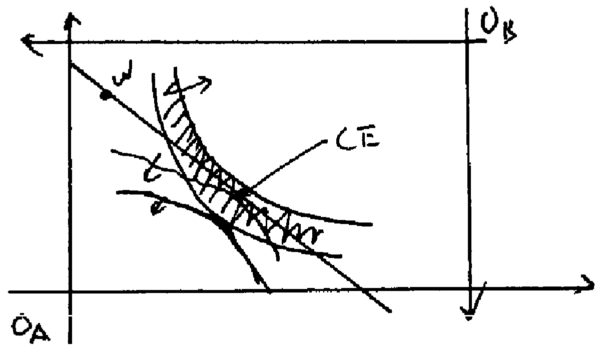

Figure 3.5: A CE allocation may not be Pareto efficient when the local non-satiation condition is not satisfied.

Remark 3.3.2 Note that neither convexity of preferences nor convexity of production set is assumed in the theorem. The conditions required for Pareto efficiency of competitive equilibrium is much weaker than the conditions for the existence of competitive equilibrium.

Remark 3.3.3 If goods are indivisible, then a competitive equilibrium allocation x may not be Pareto optimal.

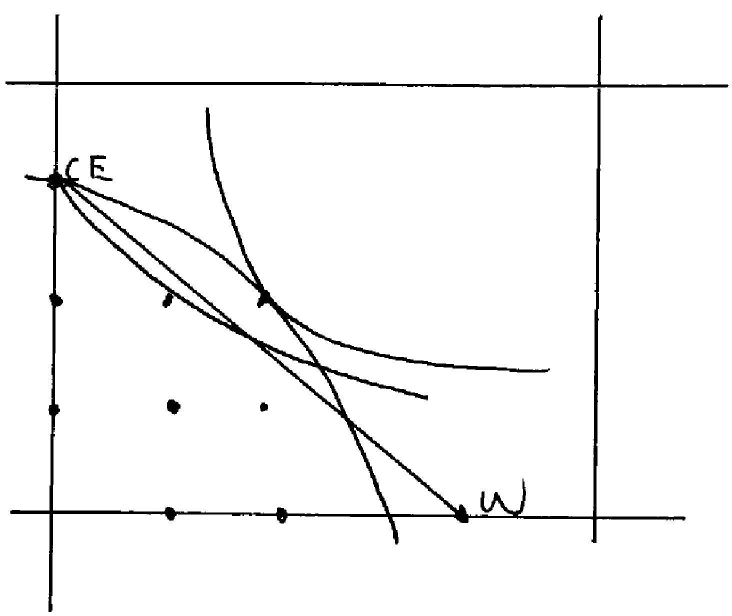

Figure 3.6: A CE allocation may not be Pareto efficient when goods are indivisible.

The point x is a competitive equilibrium allocation, but not Pareto optimal since x' is preferred by person 1. Hence, the divisibility condition cannot be dropped for the First Fundamental Theorem of Welfare Theorem to be true.

3.4 Calculations of Pareto Optimum by First-Order Conditions

The first-order conditions for Pareto efficiency are a bit harder to formulate. However, the following trick is very useful.

3.4.1 Exchange Economies

Proposition 3.4.1 *A feasible allocation x^* is Pareto efficient if and only if x^* solves the following problem for all $i = 1, 2, \ldots, n$*

$$\begin{aligned} & \max_{x_i} u_i(x_i) \\ s.t. \quad & \sum x_k \leqq \sum w_k \\ & u_k(x_k) \geqq u_k(x_k^*) \quad \text{for} \quad k \neq i \end{aligned}$$

Proof. Suppose x^* solves all maximizing problems but x^* is not Pareto efficient. This means that there is some allocation x' where one consumer is better off, and the others are not worse off. But, then x^* does not solve one of the maximizing problems, a contradiction. Conversely, suppose x^* is Pareto efficient, but it does not solve one of the problems. Instead, let x' solve that particular problem. Then x' makes one of the agents better off without hurting any of the other agents, which contradicts the assumption that x^* is Pareto efficient. The proof is completed. ■

If utility functions $u_i(x_i)$ are differentiable and x^* is an interior solution, then we can define the Lagrangian function to get the optimal solution to the above problem:

$$L = u_i(x_i) + q(\hat{w} - \hat{x}) + \sum_{k \neq i} t_k[u_k(x_k) - u_k(x_k^*)]$$

The first order conditions are then given by

$$\frac{\partial L}{\partial x_i^l} = \frac{\partial u_i(x_i)}{\partial x_i^l} - q^l = 0 \quad l = 1, 2, \ldots, L, i = 1, \ldots, n, \tag{3.6}$$

$$\frac{\partial L}{\partial x_k^l} = t_k \frac{\partial u_k(x_k)}{\partial x_k^l} - q^l = 0 \quad l = 1, 2, \ldots, L; k \neq L \tag{3.7}$$

By (3.6)

$$\frac{\frac{\partial u_i(x_i)}{\partial x_i^l}}{\frac{\partial u_i(x_i)}{\partial x_i^h}} = \frac{q^l}{q^h} = MRS_{x_i^l, x_i^h} \tag{3.8}$$

By (3.7)

$$\frac{\frac{\partial u_k(x_k)}{\partial x_k^l}}{\frac{\partial u_k(x_k)}{\partial x_k^h}} = \frac{q^l}{q^h} \tag{3.9}$$

Thus, we have

$$MRS_{x_1^l, x_1^h} = \cdots = MRS_{x_n^n, x_n^h} \quad l = 1, 2, \ldots, L; h = 1, 2, \ldots, L. \tag{3.10}$$

which are the necessary conditions for the interior solutions to be Pareto efficient, which means that the MRS of any two goods are all equal for all agents. They become sufficient conditions when utility functions $u_i(x_i)$ are differentiable and quasi-concave.

3.4.2 Production Economies

For simplicity, we assume there is only one firm. Let $T(y) \leqq 0$ be the transformation frontier. Similarly, we can prove the following proposition.

Proposition 3.4.2 *A feasible allocation (x^*, y^*) is Pareto efficient if and only if (x^*, y^*) solves the following problem for all $i = 1, 2, \ldots, n$*

$$\begin{aligned} & \max_{x_i} u_i(x_i) \\ s.t. \quad & \sum_{k \in N} x_k = \sum_{k \in N} w_k + y \\ & u_k(x_k) \geqq u_k(x_k^*) \quad for \quad k \neq i \\ & T(y) \leqq 0. \end{aligned}$$

If utility functions $u_i(x_i)$ are differentiable and x^* is an interior solution, then we can define the Lagrangian function to get the first order conditions:

$$L = u_i(x_i) + \lambda T(\hat{w} - \hat{x}) + \sum_{k \neq i} t_k [u_k(x_k) - u_k(x_k^*)]$$

FOC:

$$\begin{aligned}\frac{\partial L}{\partial x_i^l} &= \frac{\partial u_i(x_i)}{\partial x_i^l} - \lambda^l \frac{\partial T(y)}{\partial x_i^l} = 0 \quad l = 1, 2, \ldots, L, i = 1, \ldots, n, && (3.11)\\ \frac{\partial L}{\partial x_k^l} &= t_k \frac{\partial u_k(x_k)}{\partial x_k^l} - \lambda^l \frac{\partial T(y)}{\partial x_i^l} = 0 \quad l = 1, 2, \ldots, L; k \neq L \quad . && (3.12)\end{aligned}$$

By (3.11)

$$\frac{\frac{\partial u_i(x_i)}{\partial x_i^l}}{\frac{\partial u_i(x_i)}{\partial x_i^h}} = \frac{\frac{\partial T(y)}{\partial y^l}}{\frac{\partial T(y)}{\partial y^h}}. \tag{3.13}$$

By (3.12)

$$\frac{\frac{\partial u_k(x_k)}{\partial x_k^l}}{\frac{\partial u_k(x_k)}{\partial x_k^h}} = \frac{\frac{\partial T(y)}{\partial y^l}}{\frac{\partial T(y)}{\partial y^h}}. \tag{3.14}$$

Thus, we have

$$MRS_{x_1^l, x_1^h} = \cdots = MRS_{x_n^l, x_n^h} = MRTS_{y^l, y^h} \quad l = 1, 2, \ldots, L; h = 1, 2, \ldots, L \tag{3.15}$$

which are the necessary condition for the interior solutions to be Pareto efficient, which means that the MRS of any two goods for all agents equals the $MRTS$. They become sufficient conditions when utility functions $u_i(x_i)$ are differentiable and quasi-concave and the production functions are concave.

3.5 The Second Fundamental Theorem of Welfare Economics

Now we can assert a converse of the First Fundamental Theorem of Welfare Economics. The Second Fundamental Theorem of Welfare Economics gives conditions under which a Pareto optimum allocation can be "supported" by a competitive equilibrium if we allow some redistribution of endowments. It tells us, under its assumptions, including essential condition of convexity of preferences and production sets, that any desired Pareto optimal allocation can be achieved as a market-based equilibrium with transfers.

We first define the competitive equilibrium with transfer payments (also called an equilibrium relative to a price system), that is, a competitive equilibrium is established after transferring some of initial endowments between agents.

Definition 3.5.1 (Competitive Equilibrium with Transfer Payments) For an economy, $e = (e_1, \ldots, e_n, \{Y_j\})$, $(x, y, p) \in X \times Y \times R^L_+$ is a competitive equilibrium with transfer payment if

(i) $x_i \succcurlyeq_i x'_i$ for all $x'_i \in \{x'_i \in X_i : px'_i \leqq px_i\}$ for $i = 1, \ldots, n$.

(ii) $py_j \geqq py'_j$ for $y'_j \in Y_j$.

(iii) $\hat{x} \leqq \hat{w} + \hat{y}$ (feasibility condition).

Remark 3.5.1 An equilibrium with transfer payments is different from a competitive equilibrium with respect to a budget constrained utility maximization that uses a value calculated from the initial endowment. An equilibrium with transfer payments is defined without reference to the distribution of initial endowment, given the total amount.

The following theorem which is called the Second Fundamental Theorem of Welfare Economics shows that every Pareto efficient allocation can be supported by a competitive equilibrium through a redistribution of endowments so that one does not need to seek any alternative economic institution to reach Pareto efficient allocations. This is one of the most important theorems in modern economics, and the theorem is also one of the theorems in microeconomic theory that is hardest to be proved.

Theorem 3.5.1 (The Second Fundamental Theorem of Welfare Economics) *Suppose (x^*, y^*) with $x^* > 0$ is Pareto optimal, suppose $\succcurlyeq_i$ are continuous, convex and strictly monotonic, and suppose that Y_j are closed and convex. Then, there is a price vector $p \geq 0$ such that (x^*, y^*, p) is a competitive equilibrium with transfer payments, i.e.,*

(1) if $x'_i \succ_i x^*_i$, then $px'_i > px^*_i$ for $i = 1, \ldots, n$.

(2) $py^*_j \geqq py'_j$ for all $y'_j \in Y_j$ and $j = 1, \ldots, J$.

Proof: Let

$$P(x^*_i) = \{x_i \in X_i : x_i \succ_i x^*_i\} \tag{3.16}$$

be the strict upper contour set and let

$$P(x^*) = \sum_{i=1}^{n} P(x^*_i). \tag{3.17}$$

By the convexity of $\succcurlyeq_i$, we know that $P(x^*_i)$ and thus $P(x^*)$ are convex.

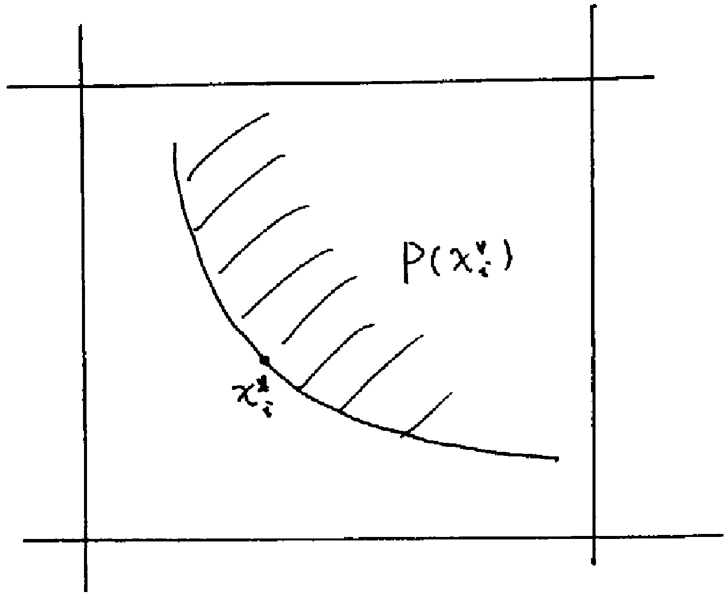

Figure 3.7: $P(x_i^*)$ is the set of all points strictly above the indifference curve through x_i^*.

Let $W = \{\hat{w}\} + \sum_{j=1}^{J} Y_j$ which is closed and convex. Then $W \cap P(x^*) = \emptyset$ by Pareto optimality of (x^*, y^*), and thus, by the Separating Hyperplane Theorem in last chapter, there is a $p \neq 0$ such that

$$p\hat{z} \geqq p\hat{\sigma} \text{ for all } \hat{z} \in P(x^*) \text{ and } \hat{\sigma} \in W \tag{3.18}$$

Now we show that p is a competitive equilibrium price vector by the following four steps.

1. $p \geqq 0$

   To see this, let $e^l = (0, \ldots, 1, 0, \ldots, 0)$ with the l-th component one and other places zero. Let

   $$\hat{z} = \hat{\sigma} + e^l \quad \text{for some } \hat{\sigma} \in W.$$

   Then $\hat{z} = \hat{\sigma} + e^l \in P(x^*)$ by strict monotonicity and redistribution of e^l. Thus, we have by (3.18).

   $$p(\hat{\sigma} + e^l) \geqq p\hat{\sigma} \tag{3.19}$$

   and thus

   $$pe^l \geqq 0 \tag{3.20}$$

   which means

   $$p^l \geqq 0 \quad for \quad l = 2, 3, \ldots, L. \tag{3.21}$$

2. $py_j^* \geqq py_j$ for all $y_j \in Y_j$, $j = 1, \ldots, J$.

   Since $\hat{x}^* = \hat{y}^* + \hat{w}$ by noting (x^*, y^*) is a Pareto efficient allocation and preference orderings are strictly monotonic, we have $p\hat{x}^* = p(\hat{w} + \hat{y}^*)$. Thus, by $p\hat{z} \geqq p(\hat{w} + \hat{y})$ in (3.18) and $p\hat{w} = p\hat{y}^* - p\hat{x}^*$, we have

$$p(\hat{z} - \hat{x}^*) \geqq p(\hat{y} - \hat{y}^*).$$

   Letting $\hat{z} \to x^*$, we have

$$p(\hat{y} - \hat{y}^*) \leqq 0.$$

   Letting $y_k = y_k^*$ for $k \neq j$, we have from the above equation,

$$py_j^* \geqq py_j \quad \forall y_j \in Y_j.$$

3. If $x_i \succ_i x_i^*$, then

$$px_i \geqq px_i^*. \tag{3.22}$$

   To see this, let

$$\begin{aligned} x_i' &= (1-\theta)x_i \quad 0 < \theta < 1 \\ x_k' &= x_k^* + \frac{\theta}{n-1}x_i \quad for \quad k \neq i \end{aligned}$$

   Then, by the continuity of $\succcurlyeq_i$ and the strict monotonicity of $\succcurlyeq_k$, we have $x_i' \succ_i x_i^*$ for all $i \in N$, and thus

$$x' \in P(x^*) \tag{3.23}$$

   if θ is sufficiently small. By (3.18), we have

$$p(x_i' + \sum_{k \neq i} x_k') = p[(1-\theta)x_i + \sum_{k \neq i}(x_k^* + \frac{\theta}{n-1}x_i)] \geqq p\sum_{k=1}^{n} x_k^* \tag{3.24}$$

   and thus we have

$$px_i \geqq px_i^* \tag{3.25}$$

4. If $x_i \succ_i x_i^*$, we must have $px_i > px_i^*$. To show this, suppose by way of contradiction, that

$$px_i = px_i^* \tag{3.26}$$

   Since $x_i \succ_i x_i^*$, then $\lambda x_i \succ_i x_i^*$ for λ sufficiently close to one by the continuity of preferences for $0 < \lambda < 1$. By step 3, we know $\lambda px_i \geqq px_i^* = px_i$ so that $\lambda \geqq 1$ by $px_i = px_i^* > 0$, which contradicts the fact that $\lambda < 1$.

■

For exchange economies, the competitive equilibrium with transfers is the same as a regular competitive equilibrium with $w_i = x_i^*$. As a corollary, we have

Corollary 3.5.1 *Suppose $x^* > 0$ is Pareto optimal, suppose $\succcurlyeq_i$ are continuous, convex and strictly monotonic. Then, x^* is a competitive equilibrium for the initial endowment $w_i = x_i^*$.*

Remark 3.5.2 If $\succcurlyeq_i$ can be represented by a concave and differentiable utility function, then the proof of the Second Fundamental Theorem of Welfare Economics can be much simpler. A sufficient condition for concavity is that the Hessian matrix is negative definite. Also note that monotonic transformation does not change preferences so that we may be able to transform a quasi-concave utility function to a concave utility function as follows, for example

$$\begin{aligned} u(x,y) &= xy \text{which is quasi-concave} \\ &\Leftrightarrow u^{\frac{1}{2}}(x,y) = x^{\frac{1}{2}} y^{\frac{1}{2}} \end{aligned}$$

which is concave after monotonic transformation.

Differentiation Version of the Second Fundamental Theorem of Welfare Economics for Exchange Economies

Proof: If $x^* > 0$ is Pareto Optimal, then we have

$$Du_i(x_i) = q/t_i \quad i = 1, 2, \ldots, n. \tag{3.27}$$

We want to show q is a competitive equilibrium price vector. To do so, we only need to show that each consumer maximizes his utility s.t. $B(p) = \{x_i \in X_i : px_i \leqq px_i^*)$. Indeed, by concavity of u_i

$$\begin{aligned} u_i(x_i) &\leqq u(x_i^*) + Du_i(x_i^*)(x_i - x_i^*) \\ &= u(x_i^*) + q(x_i - x_i^*)/t_i \\ &\leqq u(x_i^*) \end{aligned}$$

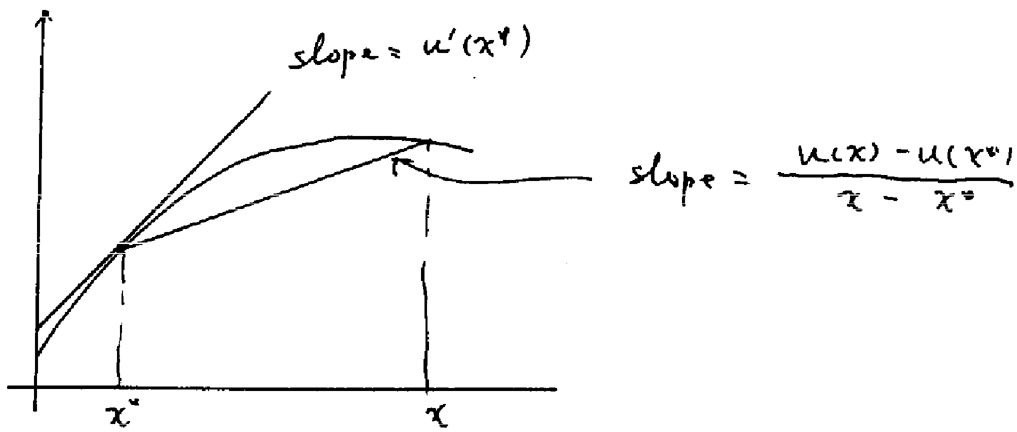

Figure 3.8: Concave funtion

The reason the inequality holds for a concave function is because that, from Figure 3.8, we have

$$\frac{u(x) - u(x^*)}{x - x^*} \leqq u'(x^*). \tag{3.28}$$

Thus, we have $u_i(x_i) \leqq u(x_i^*) + Du_i(x_i^*)\,(x_i - x_i^*)$.

3.6 Pareto Optimality and Social Welfare Maximization

Pareto efficiency is only concerned with efficiency of allocations and has nothing to say about distribution of welfare. Even if we agree with Pareto optimality, we still do not know which one we should be at. One way to solve the problem is to assume the existence of a social welfare function.

Define a social welfare function $W : X \to \Re$ by $W(u_1(x_1), \dots, u_n(x_n))$ where we assume that $W(\cdot)$ is monotone increasing.

Example 3.6.1 (The Utilitarian Social Welfare Function) $W(u_1, \dots, u_n) = \sum_{t=1}^{n} a_i u_i(x_i)$ with $\sum a_i = 1, a_i \geqq 0$. Under a utilitarian rule, social states are ranked according to the linear sum of utilities. The utilitarian form is by far the most common and widely applied social welfare function in economics.

Example 3.6.2 (The Rawlsian Social Welfare Function) $W(\cdot) = \min\{u_1(x_1), u_2(x_2), \dots, u_n(x_n)\}$. So the utility function is not strictly monotonic increasing. The Rawlsian form gives priority to the interests of the worst off members, and it is used in the ethical system proposed by Rawls (1971).

3.6.1 Social Welfare Maximization for Exchange Economies

We suppose that a society should operate at a point that maximizes social welfare; that is, we should choose an allocation x^* such that x^* solves

$$\max W(u_1(x_1), \ldots, u_n(x_n))$$

subject to

$$\sum_{i=1}^{n} x_i \leqq \sum_{i=1}^{n} w_i.$$

How do the allocations that maximize this welfare function compare to Pareto efficient allocations? The following is a trivial consequence if the strict monotonicity assumption is imposed.

Proposition 3.6.1 *Under strict monotonicity of preferences, if x^* maximizes a social welfare function, then x^* must be Pareto Optimal.*

Proof: If x^* is not Pareto Optimal, then there is another feasible allocation x' such that $u_i(x'_i) \geqq u_i(x_i)$ for all i and $u_k(x'_k) > u_k(x_k)$ for some k. Then, by strict monotonicity of preferences, we have $W(u_1(x'_1), \ldots, u_n(x'_n)) > W(u_1(x_i), \ldots, u_n(x_n))$ and thus it does not maximizes the social welfare function. ■

Thus, every social welfare maximum is Pareto efficient. Is the converse necessarily true? By the Second Fundamental Theorem of Welfare Economics, we know that every Pareto efficient allocation is a competitive equilibrium allocation by redistributing endowments. This gives us a further implication of competitive prices, which are the multipliers for the welfare maximization. Thus, the competitive prices really measure the (marginal) social value of a good. Now we state the following proposition that shows every Pareto efficient allocation is a social welfare maximum for the social welfare function with a suitable weighted sum of utilities.

Proposition 3.6.2 *Let $x^* > 0$ be a Pareto optimal allocation. Suppose u_i is concave, differentiable and strictly monotonic. Then, there exists some choice of weights a_i^* such* *that x^* maximizes the welfare functions*

$$W(u_1, \ldots, u_n) = \sum_{i=1}^{n} a_i u_i(x_i) \tag{3.29}$$

Furthermore, $a_i^* = \frac{1}{\lambda_i}$ with $\lambda_i = \frac{\partial V_i(p,I_i)}{\partial I_i}$

where $V_i(\cdot)$ is the indirect utility function of consumer i.

Proof: Since x^* is Pareto optimal, it is a competitive equilibrium allocation with $w_i = x_i^*$ by the second theorem of welfare economies. So we have

$$D_i u_i(x_i^*) = \lambda p \tag{3.30}$$

by the first order condition, where p is a competitive equilibrium price vector.

Now for the welfare maximization problem

$$\max \sum_{i=1}^{n} a_i u_i(x_i)$$
$$s.t. \sum x_i \leqq \sum x_i^*$$

since u_i is concave, x^* solves the problem if the first order condition

$$a_i \frac{\partial u_i(x_i)}{\partial x_i} = q \quad i = 1, \ldots, n \tag{3.31}$$

is satisfied for some q. Thus, if we let $p = q$, then $a_i^* = \frac{1}{\lambda_i}$. We know x^* also maximizes the welfare function $\sum_{t=1}^{n} a_i^* u_i(x_i')$. ■

Thus, the price is the Lagrangian multiples of the welfare maximization, and this measures the marginal social value of goods.

3.6.2 Welfare Maximization in Production Economy

Define a choice set by the transformation function

$$T(\hat{x}) = 0 \quad \text{with} \quad \hat{x} = \sum_{t=1}^{n} x_i. \tag{3.32}$$

The social welfare maximization problem for the production economy is

$$\max W(u_1(x_1), u_2(x_2), \ldots, u_n(u_n)) \tag{3.33}$$

subject to

$$T(\hat{x}) = 0.$$

Define the Lagrangian function

$$L = W(u_1(x_1), \dots, u_n(u_n)) - \lambda T(\hat{x}). \tag{3.34}$$

The first order condition is then given by

$$W'(\cdot)\frac{\partial u_i(x_i)}{\partial x_i^l} - \lambda\frac{\partial T(\hat{x})}{\partial x_i^l} = 0, \tag{3.35}$$

and thus

$$\frac{\frac{\partial u_i(x_i)}{\partial x_i^l}}{\frac{\partial u_i(x_i)}{\partial x_i^k}} = \frac{\frac{\partial T(\hat{x})}{\partial x^2}}{\frac{\partial T(\hat{x})}{\partial x_i^l}} \tag{3.36}$$

That is,

$$MRS_{x_i^l, x_i^k} = MRTS_{x^l, x^k}. \tag{3.37}$$

The conditions characterizing welfare maximization require that the marginal rate of substitution between each pair of commodities must be equal to the marginal rate of transformation between the two commodities for all agents.

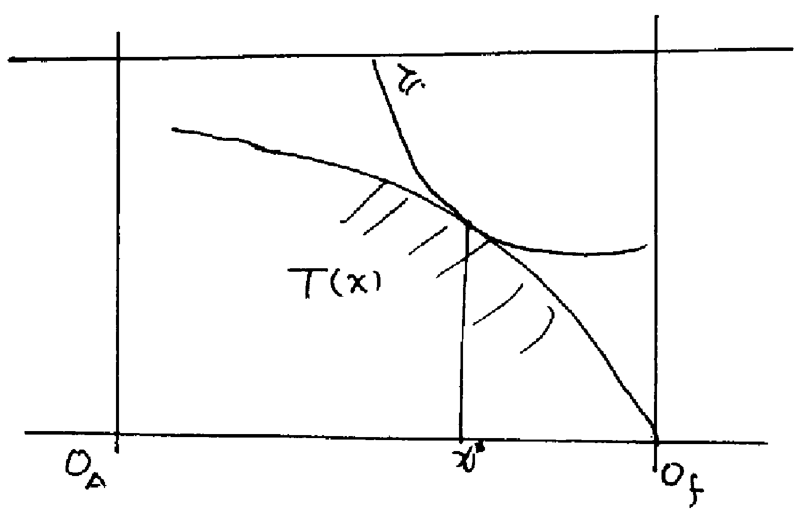

Figure 3.9: Welfare maximization

3.7 Political Overtones

1. By the First Fundamental Theorem of Welfare Economics, implication is that what the government should do is to secure the competitive environment in an economy and give

people full economic freedom. So there should be no subsidizing, no price floor, no price ceiling, stop a rent control, no regulations, lift the tax and the import-export barriers.

2. Even if we want to reach a preferred Pareto optimal outcome which may be different from a competitive equilibrium from a given endowment, you might do so by adjusting the initial endowment but not disturbing prices, imposing taxes or regulations. This is, if a derived Pareto optimal is not "fair", all the government has to do is to make a lump-sum transfer payments to the poor first, keeping the competitive environments intact. We can adjust initial endowments to obtain a desired competitive equilibrium by the Second Fundamental Theorem of Welfare Economics.

3. Of course, when we reach the above conclusions, you should note that there are conditions on the results. In many cases, we have market failures, in the sense that either a competitive equilibrium does not exist or a competitive equilibrium may not be Pareto optimal so that the First or Second Fundamental Theorem of Welfare Economics cannot be applied.

The conditions for the existence of a competitive equilibrium are: (i) convexity (diversification of consumption and no IRS), (ii) monotonicity (self-interest), and (iii) continuity, (iv) divisibility, (v) perfect competion, (vi) complete information, etc. If these conditions are not satisfied, we may not obtain the existence of a competitive equilibrium. The conditions for the First Fundamental Theorem of Welfare Economics are: (i) local non-satiation (unlimited desirability), (ii) divisibility, (iii) no externalities, (iv) perfect competion, (v) complete information etc. If these conditions are not satisfied, we may not guarantee that every competitive equilibrium allocation is Pareto efficient. The conditions for the Second Fundamental Theorem of Welfare Economics are: (i) the convexity of preferences and production sets, (ii) monotonicity (self-interest), and (iii) continuity, (iv) divisibility, (v) perfect competion, (vi) complete information, etc. If these conditions are not satisfied, we may not guarantee every Pareto efficient allocation can be supported by a competitive equilibrium with transfers.

Thus, as a general notice, before making an economic statement, one should pay attention to the assumptions which are implicit and/or explicit involved. As for a conclusion from the general equilibrium theory, one should notice conditions such as divisibility, no externalities, no increasing returns to scale, perfect competition, complete information.

If these assumptions are relaxed, a competitive equilibrium may not exist or may not be Pareto efficient, or a Pareto efficient allocation may not be supported by a competitive equilibrium with transfer payments. Only in this case of a market failure, we may adopt another economic institution. We will discuss the market failure and how to solve the market failure problem in Part II and Part III.

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Chapter 4

Economic Core, Fair Allocations, and Social Choice Theory

4.1 Introduction

In this chapter we briefly discuss some topics in the framework of general equilibrium theory, namely economic core, fair allocations, and social choice theory. Theory of core is important because it gives an insight into how a competitive equilibrium is achieved as a result of individual strategic behavior instead of results of an auctioneer and the Walrasian tâtonnement mechanism.

We have also seen that Pareto optimality may be too weak a criterion to be meaningful. It does not address any question about income distribution and equity of allocations. Fairness is a notion to overcome this difficulty. This is one way to restrict a set of Pareto optimum.

In a slightly different framework, suppose that a society is deciding the social priority among finite alternatives. Alternatives may be different from Pareto optimal allocations. Let us think of a social "rule" to construct the social ordering (social welfare function) from many individual orderings of different alternatives. The question is: Is it possible to construct a rule satisfying several desirable properties? Both "fairness" and "social welfare function" address a question of social justice.

4.2 The Core of Exchange Economies

The use of a competitive (market) system is just one way to allocate resources. What if we use some other social institution? Would we still end up with an allocation that was "close" to a competitive equilibrium allocation? The answer will be that, if we allow agents to form coalitions, the resulting allocation can only be a competitive equilibrium allocation when the economy becomes large. Such an allocation is called a core allocation and was originally considered by Edgeworth (1881).

The core is a concept in which every individual and every group agree to accept an allocation instead of moving away from the social coalition.

There is some reason to think that the core is a meaningful political concept. If a group of people find themselves able, using their own resources to achieve a better life, it is not unreasonable to suppose that they will try to enforce this threat against the rest of community. They may find themselves frustrated if the rest of the community resorts to violence or force to prevent them from withdrawing.

The theory of the core is distinguished by its parsimony. Its conceptual apparatus does not appeal to any specific trading mechanism nor does it assume any particular institutional setup. Informally, notion of competition that the theory explores is one in which traders are well informed of the economic characteristics of other traders, and in which the members of any group of traders can bind themselves to any mutually advantageous agreement.

For simplicity, we consider exchange economies. We say two agents are of the same type if they have the same preferences and endowments.

The r-replication of the original economy: There are r times as many agents of each type in the original economy.

A coalition is a group of agents, and thus it is a subset of n agents.

Definition 4.2.1 (Blocking Coalition) A group of agents S (a coalition) is said to block (improve upon) a given allocation x if there is some allocation x' such that

(1) it is feasible for S, ie., $\sum_{i \in S} x'_i \leqq \sum_{i \in S} w_i$,

(2) $x'_i \succcurlyeq_i x_i$ for all $i \in S$ and $x'_k \succ_k x_k$ for some $k \in S$.

Definition 4.2.2 (Core) A feasible allocation x is said to be in the core of an economy if it cannot be improved upon for any coalition.

Remark 4.2.1 Every core allocation is Pareto optimal (coalition by whole people).

Definition 4.2.3 (Individual Rationality) An allocation x is individually rational if $x_i \succcurlyeq_i w_i$ for all $i = 1, 2, \ldots, n$.

The individual rationality condition is also called the participation condition which means that a person will not participate the economic activity if he is worse off than at the initial endowment.

Remark 4.2.2 Every core allocation must be individually rational.

Remark 4.2.3 When $n = 2$, an allocation is in the core if and only if it is Pareto optimal and individually rational.

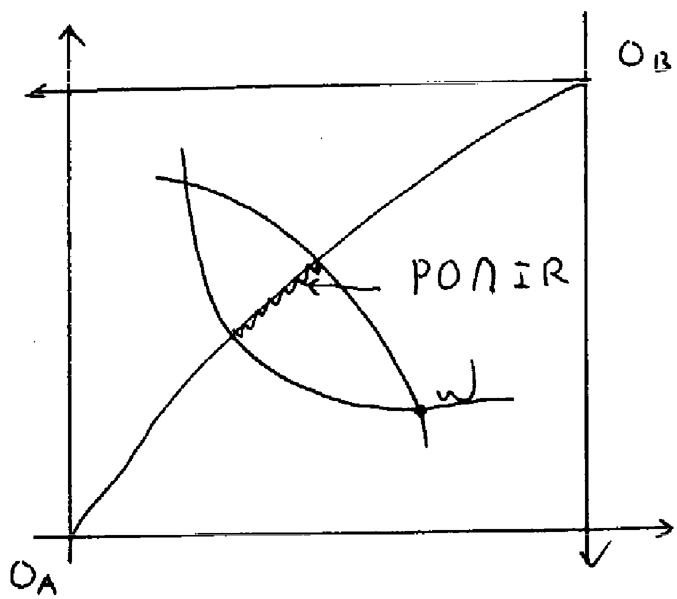

Figure 4.1: The set of core allocations are simply given by the set of Pareto efficient and individually rational allocations when $n = 2$.

Remark 4.2.4 Even though a Pareto optimal allocation is independent of individual endowments, a core allocation depends on individual endowments.

What is the relationship between core allocations and competitive equilibrium allocations?

Theorem 4.2.1 *Under local non-satiation, if (x, p) is a competitive equilibrium, then x is a core allocation.*

Proof: Suppose x is not a core allocation. Then there is a coalition S and a feasible allocation x' such that

$$\sum_{i \in S} x'_i \leqq \sum_{i \in S} w_i \tag{4.1}$$

and $x'_i \succcurlyeq_i x_i$ for all $i \in S$, $x'_k \succ_k x_k$ for some $k \in S$. Then, by local non-satiation, we have

$$\begin{aligned} px'_i &\geqq px_i \text{ for all } i \in S \text{ and} \\ px'_k &> px_k \text{ for some } k \end{aligned}$$

Therefore, we have

$$\sum_{i \in S} px'_i > \sum_{i \in S} px_i = \sum_{i \in S} w_i \tag{4.2}$$

a contradiction. Therefore, the competitive equilibrium must be a core allocation.

The following proposition is a converse of the above proposition and shows that any allocation that is not a market equilibrium allocation must eventually not be in the r-core of the economy. This means that core allocations in large economies look just like Walrasian equilibria.

Theorem 4.2.2 (Shrinking Core Theorem) *Suppose $\succcurlyeq_i$ are strictly convex, strongly monotonic, and continuous. Suppose x^* is a unique competitive equilibrium allocation. Then, if y is not a competitive equilibrium, there is some replication V such that y is not in the V-core.*

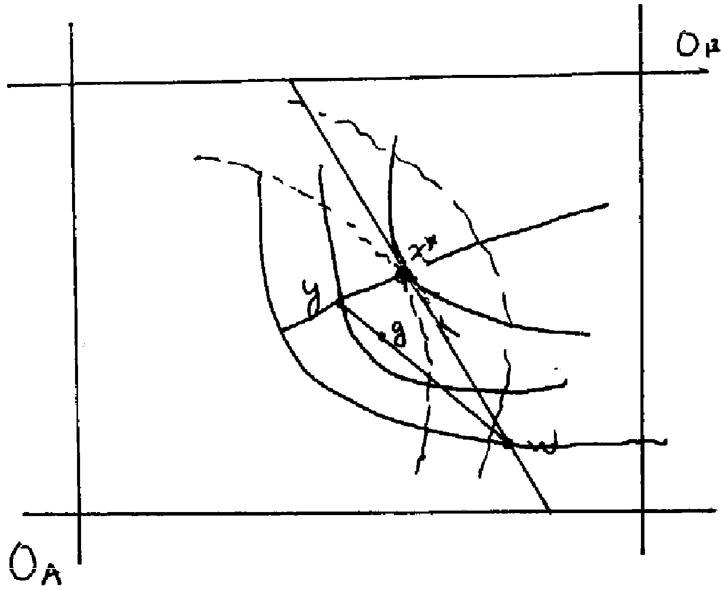

Figure 4.2: The shrinking core. A point like y will eventually not be in the core.

Proof: We want to show that there is a coalition such that the point y can be improved upon for V-replication. Since y is not a competitive equilibrium, the line segment through y and w must cut at least one agent's, say agent A's, indifference curve through y. Then, by strict convexity and continuity of $\succcurlyeq_i$, there are integers V and T with $0 < T < V$ such that

$$g_A \equiv \frac{T}{V}w_A + (1 - \frac{T}{V})y_A \succ_A y_A.$$

Form a coalition consisting of V consumers of type A and $V - T$ consumers of type B. Consider the allocation $x = (g_A, \ldots g_A,\, y_B, \ldots y_B)$(in which there are V Type A and $V - T$ Type B). We want to show x is feasible for this coalition.

$$\begin{aligned} Vg_A + (V-T)y_B &= V\left[\frac{T}{V}w_A + \left(1 - \frac{T}{V}\right)\right] + (V-T)y_B \\ &= Tw_A + (V-T)y_A + (V-T)y_B \\ &= Tw_A + (V-T)(y_A + y_B) \\ &= Tw_A + (V-T)(w_A + w_B) \\ &= Vw_A + (V-T)w_B \end{aligned}$$

by noting $y_A + y_B = w_A + w_B$. Thus, x is feasible in the coalition and $g_A \succ_A y_A$ for all agents in type A and $y_B \sim_B y_B$ for all agents in type B which means y is not in the V-core for the V-replication of the economy. The proof is completed.

Remark 4.2.5 The shrinking core theorem then shows that the only allocations that are in the core of a large economy are market equilibrium allocations, and thus Walrasian equilibria are robust: even very weak equilibrium concepts, like that of core, tend to yield allocations that are close to Walrasian equilibria for larger economies. Thus, this theorem shows the essential importance of competition and fully economic freedom.

Remark 4.2.6 Many of the restrictive assumptions in this proposition can be relaxed such as strict monotonicity, convexity, uniqueness of competitive equilibrium, and two types of agents.

From the above discussion, we have the following limit theorem.

Theorem 4.2.3 (Limit Theorem on the Core) *Under the strict convexity, strict monotonicity and continuity, the core of a replicated two person economy shrinks when the number of agents for each type increases, and the core coincides with the competitive equilibrium allocation if the number of agents goes to infinity.*

This result means that any allocation which is not a competitive equilibrium allocation is not in the core for some r-replication.

4.3 Fairness of Allocation

Pareto efficiency gives a criterion of how the goods are allocated efficiently, but it may be too weak a criterion to be meaningful. It does not address any questions about income distribution, and does not give any "equity" implication. Fairness is a notion that may overcome this difficulty. This is one way to restrict the whole set of Pareto efficient outcomes to a small set of Pareto efficient outcomes that satisfy the other properties.

What is the equitable allocation?

How can we define the notion of equitable allocation?

Definition 4.3.1 (Envy) An agent i is said to envy agent k if agent i prefers agent k's consumption. i.e., $x_k \succ_i x_i$.

Definition 4.3.2 An allocation x is equitable if no one envies anyone else, i.e., for each $i \in N$, $x_i \succcurlyeq_i x_k$ for all $k \in N$.

Definition 4.3.3 (Fairness) An allocation x is said to be fair if it is both Pareto optimal and equitable.

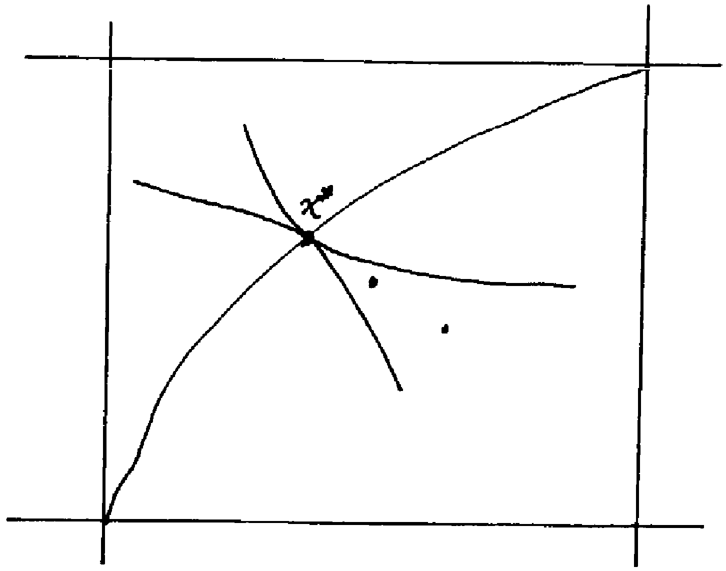

Figure 4.3: A fair allocation.

Remark 4.3.1 By the definition, a set of fair allocations is a subset of that of Pareto efficient allocations. Therefore, fairness restricts the size of Pareto optimal allocations.

The following strict fairness concept is due to Lin Zhou (JET, 1992, 57: 158-175).

An agent i envies a coalition S ($i \notin S$) at an allocation x if $\bar{x}_S \succ_i x_i$, where $\bar{x}_S = \frac{1}{|S|}\sum_{j \in S} x_j$.

Definition 4.3.4 An allocation x is strictly equitable or strictly envy-free if no one envies any other coalitions.

Definition 4.3.5 (Strict Fairness) An allocation x is said to be strictly fair if it is both Pareto optimal and strictly equitable.

Remark 4.3.2 A set of strictly fair allocation S are a subset of Pareto optimal allocations.

Remark 4.3.3 For a two person exchange economy, if x is Pareto optimal, it is impossible for two persons to envy each other.

Remark 4.3.4 It is clear every strictly fair allocation is a fair allocation, but the converse may not be true. However, when $n = 2$, a fair allocation is a strictly fair allocation.

The following figure shows that x is Pareto efficient, but not equitable.

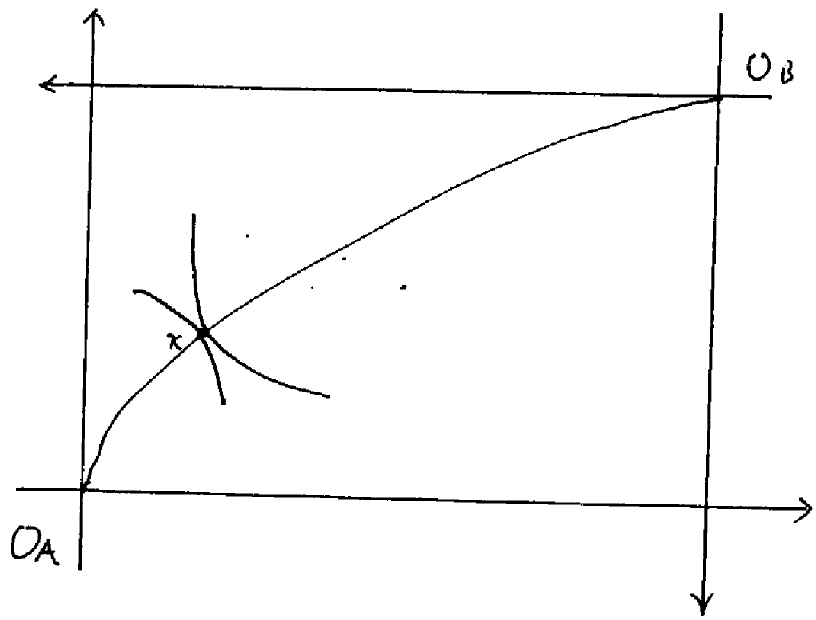

Figure 4.4: x is Pareto efficient, but not equitable.

The figure below shows that x is equitable, but it is not Pareto efficient.

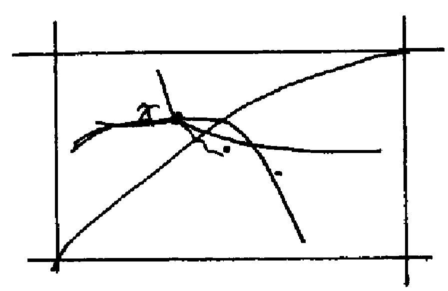

Figure 4.5: x is equitable, but not Pareto efficient.

How to test a fair allocation?

Graphical Procedure for Testing Fairness:

Let us restrict an economy to a two-person economy. An easy way for agent A to compare his own allocation x_A with agent B's allocation x_B in the Edgeworth Box is to find a point symmetric of x_A against th center of the Box. That is, draw a line from x_A to the center of the box and extrapolate it to the other side by the same length to find

x'_A, and then make the comparison. If the indifference curve through x_A cuts "below" x'_A, then A envies B. Then we have the following way to test whether an allocation is a fair allocation:

Step 1: Is it Pareto optimality? If the answer is "yes", go to step 2; if no, stop.

Step 2: Construct a reflection point (x_B, x_A). (Note that $\frac{x_A+x_B}{2}$ is the center of the Edgeworth box.)

Step 3: Compare x_B with x_A for person A to see if $x_B \succ_A x_A$ and compare x_A with x_B for person B to see if $x_A \succ_B x_B$. If the answer is "no" for both persons, it is a fair allocation.

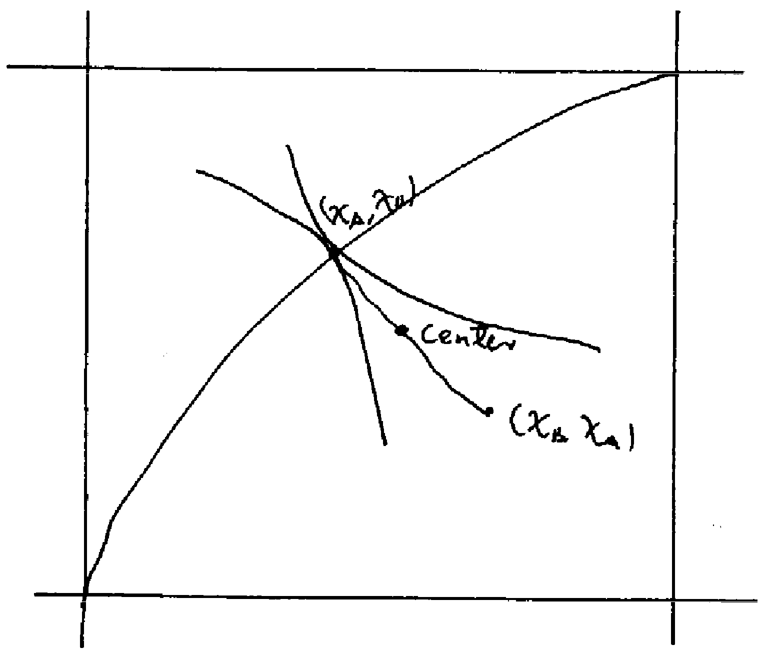

Figure 4.6: How to test a fair allocation.

We have given some desirable property of "fair" allocation. A question is whether it exists at all. The following theorem provides one sufficient condition to guarantee the existence of fairness.

Theorem 4.3.1 *Let (x^*, p^*) be a competitive equilibrium. Under local non-satiation, if all individuals' income is the same, i.e., $p^*w_1 = p^*w_2 = \ldots = p^*w_n$, then x^* is a strictly fair allocation.*

Proof: By local non-satiation, x^* is Pareto optimal by the First Fundamental Theorem of Welfare Economics. We only need to show x^* is strictly equitable. Suppose not. There is i and a coalition S with $i \notin S$ such that

$$\overline{x}^*_S \equiv \frac{1}{|S|}\sum_{k\in S} x^*_k \succ_i x^*_i \tag{4.3}$$

Then, we have $p^*\overline{x}^*_S > p^*x^*_i = p^*w_i$. But this contradicts the fact that

$$p^*\overline{x}^*_S = \frac{1}{|S|}\sum_{k\in S} p^*x^*_k = \frac{1}{|S|}\sum_{k\in S} p^*w_k = p^*w_i \tag{4.4}$$

by noting that $p^*w_1 = p^*w_2 = \ldots = p^*w_n$. Therefore, it must be a strictly fair allocation.

Definition 4.3.6 An allocation $x \in \Re^{nL}_+$ is an equal income Walrasian allocation if there exists a price vector such that

(1) $px_i \leqq p\overline{w}$, where $\overline{w} = \frac{1}{n}\sum_{k=1}^n w_k$: average endowment.

(2) $x'_i \succ_i \ x_i$ implies $px'_i > px_i$

(3) $\sum x_i \leqq \sum w_i$

Notice that every equal Walrasian allocation x is a competitive equilibrium allocation with $w_i = \overline{w}$ for all i.

Corollary 4.3.1 *Under local non-satiation, every equal income Walrasian allocation is strictly fair allocation.*

Remark 4.3.5 An "equal" division of resource itself does not give "fairness," but trading from "equal" position will result in "fair" allocation. This "divide-and-choose" recommendation implies that if the center of the box is chosen as the initial endowment point, the competitive equilibrium allocation is fair. A political implication of this remark is straightforward. Consumption of equal bundle is not Pareto optimum, if preferences are different. However, an equal division of endowment plus competitive markets result in fair allocations.

Remark 4.3.6 A competitive equilibrium from an equitable (but not "equal" division) endowment is not necessarily fair.

Remark 4.3.7 Fair allocation are defined without reference to initial endowment. Since we are dealing with optimality concepts, initial endowments can be redistributed among agents in a society.

In general, there is no relationship between core and fair allocations. However, when the social endowments are divided equally among two persons, we have the following theorem.

Theorem 4.3.2 *In a two-person exchange economy, if $\succcurlyeq_i$ are convex, and if the total endowments are equally divided among individuals, then the set of core allocations is a subset of (strictly) fair allocation.*

Proof: We know that a core allocation x is Pareto optimal. We only need to show that x is equitable. Since x is a core allocation, then x is individually rational (everyone prefers initial endowments). If x is not equitable, there is some agent i, say, agent A, such that

$$\begin{aligned} x_B \succ_A x_A \succcurlyeq_A w_A &= \frac{1}{2}(w_A + w_B) \\ &= \frac{1}{2}(x_A + x_B) \end{aligned}$$

by noting that $w_A = w_B$ and x is feasible. Thus, $x_A \succcurlyeq_A \frac{1}{2}(x_A + x_B)$. But, on the other hand, since $\succcurlyeq_A$ is convex, $x_B \succ_A x_A$ implies that $\frac{1}{2}(x_A + x_B) \succ_A x_A$, a contradiction.

4.4 Social Choice Theory

4.4.1 Introduction

In this section, we present a vary brief summary and introduction of social choice theory. We analyze the extent to which individual preferences can be aggregated into social preferences, or more directly into social decisions, in a "satisfactory" manner (in a manner compatible with the fulfilment of a variety of desirable conditions.

As was shown in the discussion of "fairness," it is difficult to come up with a criterion (or a constitution) that determines that society's choice. Social choice theory aims at constructing such a rule which could be allied with not only Pareto efficient allocations, but also any alternative that a society faces. We will give some fundamental results of social choice theory: Arrow Impossibility Theorem, which states there does not exist any non-dictatorial social welfare function satisfying a number of "reasonable" assumptions. Gibbard-Satterthwaite theorem states that no social choice mechanism exists which is non-dictatorial and can never be advantageously manipulated by some agents.

4.4.2 Basic Settings

$N = \{1, 2, ...n\}$: the set of individuals;

$X = \{x_1, x_2, ...x_m\}$: $(m \geqq 3)$: the set of alternatives (outcomes);

$P_i =$ (or $\succ_i$) : strict preference orderings of agent i;

$\sum(X) =$ the class of allowed individual orderings;

$P = (P_1, P_2, ..., P_n)$: a preference ordering profile;

$[\sum(X)]^n$: the set of all profiles of individuals orderings;

$S(X)$: the class of allowed social orderings.

Arrow's social welfare function:

$$F : [\sum(X)]^n \to S(X) \tag{4.5}$$

which is a mapping from individual ordering profiles to social orderings.

Gibbard-Satterthwaite's social choice function (SCF) is a mapping from individual preference orderings to the alternatives

$$f : [\sum(X)]^n \to X \tag{4.6}$$

Note that even though individuals' preference orderings are transitive, a social preference ordering may not be transitive. To see this, consider the following example.

Example 4.4.1 (The Condorect Paradox) Suppose a social choice is determined by the majority voting rule. Does this determine a social welfare function? The answer is in general no by the well-known Condorect paradox. Consider a society with three agents and three alternatives: x, y, z. Suppose each person's preference is given by

$x \succ_1 y \succ_1 z$ (by person 1)

$y \succ_2 z \succ_2 x$ (by person 2)

$z \succ_3 x \succ_3 y$ (by person 3)

By the majority rule,

For x and y, xFy (by social preference)

For y and z, yFz (by social preference)

For x and z, zFx (by social preference)

Then, pairwise majority voting tells us that x must be socially preferred to y, y must be socially preferred to z, and z must be socially preferred to x. This cyclic pattern means that social preference is not transitive.

The number of preference profiles can increase very rapidly with increase of number of alternatives.

Example 4.4.2 $X = \{x, y, z\}$, $n = 3$

$x \succ y \succ z$

$x \succ z \succ y$

$y \succ x \succ z$

$y \succ z \succ x$

$z \succ x \succ y$

$z \succ y \succ x$

Thus, there are six possible individual orderings, i.e., $|\sum(X)| = 6$, and therefore there are $|\sum(X)|^3 = 6^3 = 216$ possible combinations of 3-individual preference orderings on three alternatives. The social welfare function is a mapping from each of these 216 entries (cases) to one particular social ordering (among six possible social orderings of three alternatives). The social choice function is a mapping from each of these 216 cases to one particular choice (among three alternatives). A question we will investigate is what kinds of desirable conditions should be imposed on these social welfare or choice functions.

You may think of a hypothetical case that you are sending a letter of listing your preference orderings, P_i, on announced national projects (alternatives), such as reducing deficits, expanding the medical program, reducing society security program, increasing national defence budget, to your Congressional representative. The Congress convenes with a hug stake of letters P and try to come up with national priorities $F(P)$. You want the Congress to make a certain rule (the Constitution) to form national priorities out of individual preference orderings. This is a question addressed in social choice theory.

4.4.3 Arrow's Impossibility Theorem

Definition 4.4.1 Unrestricted Domain (UD): A class of allowed individual orderings $\sum(X)$ consists of all possible orderings defined on X.

Definition 4.4.2 Pareto Principle (P): if for $x, y \in X$, $xP_i\, y$ for all $i \in N$, then $xF(P)y$ (social preferences).

Definition 4.4.3 Independence of Irrelevant Alternatives (IIA): For $x, y, \in X$, $P, P' \in [\Sigma(X)]^n$ "xP_iy if and only if xP'_iy for all $i \in N$" implies that $xF(P)y$ if and only if $xF(P')y$.

Remark 4.4.1 IIA means that the ranking between x and y for any agent is equivalent in terms of P and P' implies that the social ranking between x and y by $F(P)$ and $F(P')$ is the same. IN other words, if two different preference profiles that are the same on x and y, the social order must also the same on x and y.

Remark 4.4.2 By IIA, any change in preference ordering other than the ordering of x and y should not affect social ordering between x and y.

Example 4.4.3 Suppose $x\ P_i\ y\ P_i\ z$ and $x\ P'_i\ z\ P'_i\ y$.

By IIA, if $xF(P)y$, then $xF(P')y$.

Definition 4.4.4 (Dictator) There is some agent $i \in N$ such that $F(P) = P_i$ for all $P \in [\Sigma(X)]^n$, and agent i is called a dictator.

Theorem 4.4.1 (Arrow's Impossibilities Theorem) *Any social welfare function that satisfies $m \geqq 3$, UD, P, IIA conditions is dictatorial.*

The proof of Arrow's Impossibility Theorem is very complicated, and the readers who are interested in the proof are referred to Mas-Colell, Whinston, and Green (1995).

The impact of the Arrow possibility theorem has been quite substantial. Obviously, Arrow's impossibility result is a disappointment. The most pessimistic reaction to it is to conclude that there is just no acceptable way to aggregate individual preferences, and hence no theoretical basis for treating welfare issues. A more moderate reaction, however, is to examine each of the assumptions of the theorem to see which might be given up. Conditions imposed on social welfare functions may be too restrictive. Indeed, when some conditions are relaxed, then the results could be positive, e.g., UD is usually relaxed.

4.4.4 Some Positive Result: Restricted Domain

When some of the assumptions imposed in Arrow's impossibility theorem is removed, the result may be positive. For instance, if alternatives have certain characteristics which could be placed in a spectrum, preferences may show some patterns and may not exhaust all possibilities orderings on X, thus violating (UD). In the following, we consider the case of restricted domain. A famous example is a class of "single-peaked" preferences. We show that under the assumption of single-peaked preferences, non-dictatorial aggregation is possible.

Definition 4.4.5 A binary relation $\geq$ on X is a *linear order* if $\geq$ is reflexive ($x \geq x$), transitive ($x \geq y \geq z$ implies $x \geq z$), and total (for distinct $x, y \in X$, either $x \geq y$ or $y \geq x$, but not both).

Example: $X = \Re$ and $x \geqq y$.

Definition 4.4.6 $\succcurlyeq_i$ is said to be *single-peaked* with respect to the linear order $\geq$ on X, if there is an alternative $x \in X$ such that $\succcurlyeq_i$ is increasing with respect to $\geq$ on the lower contour set $L(x) = \{y \in X : y \leq x\}$ and decreasing with respect to $\geq$ on the upper contour set $U(x) = \{y \in X : y \geq x\}$. That is,

(1) $x \geq z > y$ implies $z \succ_i y$

(2) $y > z \geq x$ implies $z \succ_i y$

In words, there is an alternative that represents a peak of satisfaction and, moreover, satisfaction increases as we approach this peak so that, in particular, there cannot be any other peak of satisfaction.

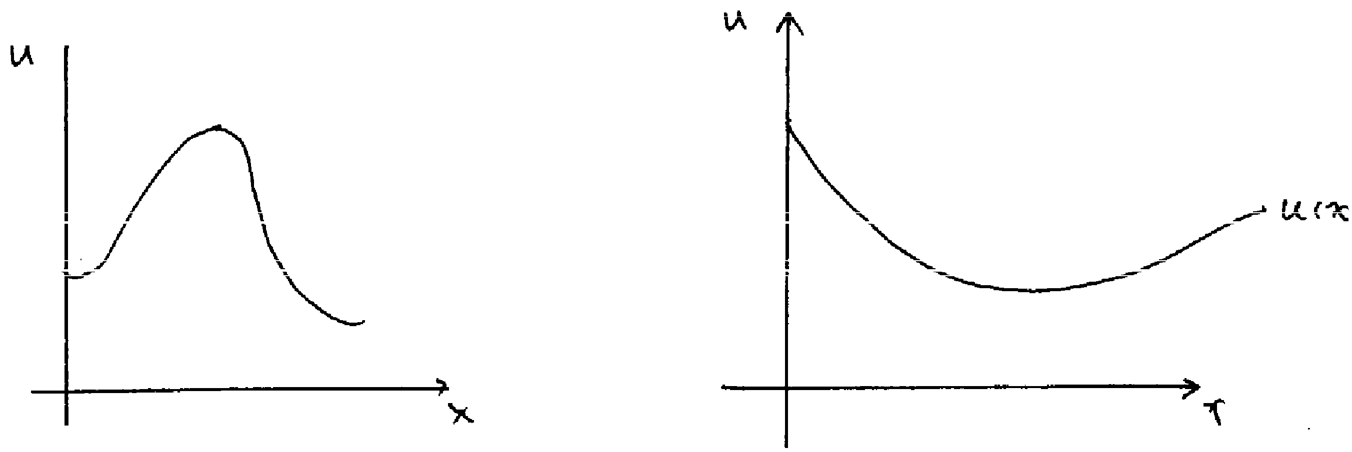

Figure 4.7: u in the left figure is single-peaked, u in the right figure is not single-peaked.

Given a profile of preference $(\succsim_1, \ldots, \succsim_n)$, let x_i be the maximal alternative for $\succsim_i$ (we will say that x_i is "individual i's peak").

Definition 4.4.7 Agent $h \in N$ is a median agent for the profile $(\succsim_1, \ldots, \succsim_n)$ if $\#\{i \in N : x_i \geq x_h\} \geq \frac{I}{2}$ and $\#\{i \in N : x_h \geq x_i\} \geq \frac{I}{2}$.

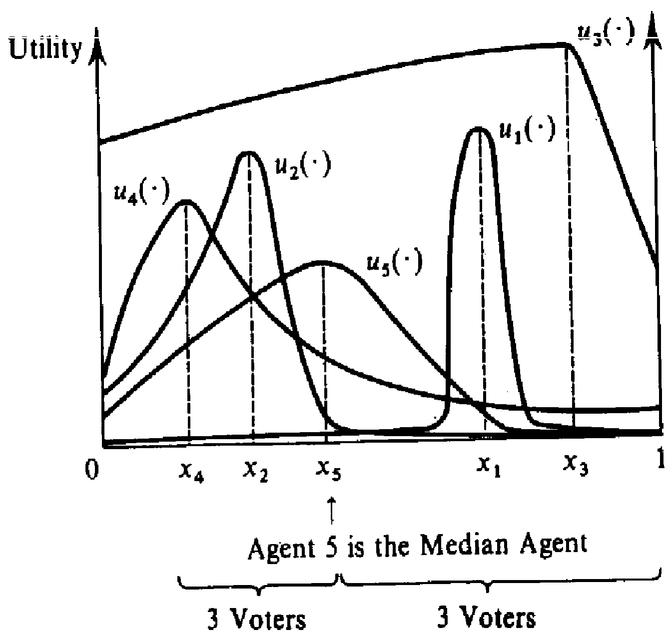

Figure 4.8: Five Agents who have single-peaked preferences.

Proposition 4.4.1 *Suppose that $\geq$ is a linear order on X, $\succsim_i$ is single-peaked. Let $h \in N$ be a median agent, then the majority rule $\tilde{F}(\succsim)$ is aggregatable:*

$$x_h \tilde{F}(\succsim) y \forall y \in X.$$

That is, the peak x_h of the median agent is socially optimal (cannot be defeated by any other alternative) by majority voting. Any alternative having this property is called a Condorect winner. Therefore, a Condorect winner exists whenever the preferences of all agents are single-peaked with respect to the same linear order.

Proof. Take any $y \in X$ and suppose that $x_h > y$ (the argument is the same for $y > x_h$). We need to show that

$$\#\{i \in N : x_h \succ_i y\} \geq \#\{i \in N : y \succ_i x_h\}.$$

Consider the set of agents $S \subset N$ that have peaks larger than or equal to x_h, that is, $S = \{i \in N : x_i \geq x_h\}$. Then $x_i \geq x_h > y$ for every $i \in S$. Hence, by single-peakness of $\succcurlyeq_i$ with respect to $\geq$, we get $x_h \succ_i y$ for every $i \in S$. On the other hand, because agent h is a median agent, we have that $\#S \geq n/2$ and so $\#\{i \in N : y \succ_i x_h\} \leqq \#(N \setminus S) \leqq n/2 \leqq \#S \leqq \#\{i \in N : x_h \succ_i y\}$.

4.4.5 Gibbard-Satterthwaite Impossibility Theorem

Definition 4.4.8 A social choice function (SCF) is manipulable at $P \in [\Sigma(X)]^n$ if there exists $P'_i \in \Sigma(X)$ such that

$$f(P_{-i}, P'_i)\ P_i\ f(P_{-i}, P_i) \tag{4.7}$$

where $P_{-i} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)$.

Definition 4.4.9 A SCF is strongly individually incentive compatible (SIIC) if there exists no preference ordering profile at which it is manipulable. In other words, the truth telling is a dominant strategy equilibrium:

$$f(P'_{-i}, P_i) P_i f(P'_i, P'_i) \quad \text{for all } P' \in [\Sigma(X)]^n \tag{4.8}$$

Definition 4.4.10 A SCF is dictatorial if there exists an agent whose optimal choice is the social optimal.

Theorem 4.4.2 (Gibbard-Satterthwaite Theorem) *If X has at least 3 alternatives, a SCF which is SIIC and UD is dictatorial.*

Again, the proof of Gibbard-Satterthwaite's Impossibility Theorem is very complicated, and the readers who are interested in the proof are referred to Mas-Colell, Whinston, and Green (1995).

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Part II

Externalities and Public Goods

In Chapters 2 and 3, we have introduced the notions of competitive equilibrium and Pareto optimality, respectively. The concept of competitive equilibrium provides us with an appropriate notion of market equilibrium for competitive market economies. The concept of Pareto optimality offers a minimal and uncontroversial test that any social optimal economic outcome should pass since it is a formulation of the idea that there is further improvement in society, and it conveniently separates the issue of economic efficiency from more controversial (and political) questions regarding the ideal distribution of well-being across individuals.

The important results and insights we obtained in these chapters are the First and Second Fundamental Theorems of Welfare Economics. The first welfare theorem provides a set of conditions under which we can be assured that a market economy will achieve a Pareto optimal; it is, in a sense, the formal expression of Adam Smith's claim about the "invisible hand" of the market. The second welfare theorem goes even further. It states that under the same set of conditions as the first welfare theorem plus convexity and continuity conditions, all Pareto optimal outcomes can in principle be implemented through the market mechanism by appropriately redistributing wealth and then "letting the market work."

Thus, in an important sense, the general equilibrium theory establishes the perfectly competitive case as a benchmark for thinking about outcomes in market economies. In particular, any inefficiency that arise in a market economy, and hence any role for Pareto-improving market intervention, must be traceable to a violation of at least one of these assumptions of the first welfare theorem. The remainder of this course, can be viewed as a development of this theme, and will study a number of ways in which actual markets may depart from this perfectly competitive ideal and where, as a result, market equilibria fail to be Pareto optimal, a situation known market failure.

In the current part, we will study externalities and public goods in Chapter 5 and Chapter 6, respectively. In both cases, the actions of one agent directly affect the utility or production of other agents in the economy. We will see these nonmarketed "goods" or "bads" lead to a non-Pareto optimal outcome in general; thus a market failure. It turns out that private markets are often not a very good mechanism in the presence of externalities and public goods. We will consider situations of incomplete information which also result in non-Pareto optimal outcomes in general in Part III.

Chapter 5

Externalities

5.1 Introduction

In this chapter we deal with the case of externalities so that a market equilibrium may lead to non-Pareto efficient allocations in general, and thus there is a market failure. The reason is that there are things that people care about are not priced. Externality can happen in both cases of consumption and production.

Consumption Externality:

$$u_i(x_i) : \text{without preference externality}$$
$$u_i(x_1, ..., x_n) : \text{with preference externality}$$

in which other individuals's consumption enters the person's utility function.

Example 5.1.1 (i) One person's quiet environment is disturbed by another person's local stereo.

(ii) Mr. A hates Mr. D smoking next to him.

(ii) Mr. A's satisfaction decreases as Mr. C's consumption level increases, because Mr. A envies Mr. C's lifestyle.

Production Externality:

A firm's production includes arguments other than its own inputs.

For example, downstream fishing is adversely affected by pollutants emitted from an upstream chemical plant.

This leads to an examination of various suggestions for alternative ways to allocate resources that may lead to efficient outcomes. Achieving an efficient allocation in the presence of externalities essentially involves making sure that agents face the correct prices for their actions. Ways of solving externality problem include taxation, regulation, property rights, merges, etc.

5.2 Consumption Externalities

When there are no consumption externalities, agent i's utility function is a function of only his own consumption:

$$u_i(x_i) \tag{5.1}$$

In this case, the first order conditions for the competitive equilibrium are given by

$$MRS^A_{xy} = MRS^B_{xy} = \frac{p_x}{p_y}$$

and the first order conditions for Pareto efficiency are given by:

$$MRS^A_{xy} = MRS^B_{xy}.$$

So, every competitive equilibrium implies Pareto efficiency if utility functions are quasi-concave.

The main purpose of this section is to show that a competitive equilibrium allocation is not in general Pareto efficient when there exists an externality in consumption. We show this by examining that the first order conditions for a competitive equilibrium is not in general the same as the first order conditions for Pareto efficient allocations in the presence of consumption externalities.

Consider the following simple two-person and two-good exchange economy.

$$u_A(x_A, x_B, y_A) \tag{5.2}$$

$$u_B(x_A, x_B, y_B) \tag{5.3}$$

which are assumed to be strictly increasing, quasi-concave, and satisfies the Inada condition $\frac{\partial u}{\partial x_i}(0) = +\infty$ and $\lim_{x_i \to 0} \frac{\partial u}{\partial x_i} x_i = 0$ so it results in interior solutions. Here good x results in consumption externalities.

The first order conditions for the competitive equilibrium are the same as before:

$$MRS^A_{xy} = \frac{p_x}{p_y} = MRS^B_{xy}.$$

To find the first order conditions for Pareto efficient allocations, we now solve the following problem

$$\max_{(x,y)} \quad a_A u_A(x_A, x_B, y_A) + a_B u_B(x_A, x_B, y_B) \tag{5.4}$$

$$s.t. \quad x_A + x_B = \hat{w}_x \tag{5.5}$$

$$y_A + y_B = \hat{w}_y \tag{5.6}$$

Define

$$L = a_A u_A(x_A, x_B, y_B) + a_B u_B(x_A, x_B, y_B) + \lambda(\hat{w}_x - x_A - x_B) + \mu(\hat{w}_y - y_A - y_B) \tag{5.7}$$

The first order conditions for interior solutions are:

$$x_A: \quad a_A \frac{\partial u_A}{\partial x_A} + a_B \frac{\partial u_B}{\partial x_A} - \lambda = 0 \tag{5.8}$$

$$x_B: \quad a_A \frac{\partial u_B}{\partial x_A} + a_B \frac{\partial u_B}{\partial x_B} - \lambda = 0 \tag{5.9}$$

$$y_A: \quad a_A \frac{\partial u_A}{\partial y_A} - \mu = 0 \tag{5.10}$$

$$y_B: \quad a_B \frac{\partial u_B}{\partial x_B} - \mu = 0 \tag{5.11}$$

Substituting (5.10) and (5.11) into (5.8) and (5.9), we have

$$\frac{\frac{\partial u_A}{\partial x_A}}{\frac{\partial u_A}{\partial y_A}} + \frac{\frac{\partial u_B}{\partial x_A}}{\frac{\partial u_B}{\partial y_B}} = \frac{\lambda}{\mu} \tag{5.12}$$

$$\frac{\frac{\partial u_A}{\partial x_B}}{\frac{\partial u_A}{\partial y_A}} + \frac{\frac{\partial u_B}{\partial x_B}}{\frac{\partial u_B}{\partial y_B}} = \frac{\lambda}{\mu} \tag{5.13}$$

and thus

$$\frac{\frac{\partial u_A}{\partial x_A}}{\frac{\partial u_A}{\partial y_A}} + \frac{\frac{\partial u_B}{\partial x_A}}{\frac{\partial u_B}{\partial y_B}} = \frac{\frac{\partial u_A}{\partial x_B}}{\frac{\partial u_A}{\partial y_A}} + \frac{\frac{\partial u_B}{\partial x_B}}{\frac{\partial u_B}{\partial y_B}} \tag{5.14}$$

That is,

$$MRS^A_{x_A y_A} - MRS^A_{x_B y_A} = MRS^B_{x_B y_B} - MRS^B_{x_A y_B}. \tag{5.15}$$

which is different from the first order conditions for competitive equilibrium: $MRS^A_{x_B y_A} = MRS^B_{x_B y_B}$. So the competitive equilibrium allocations may not be Pareto optimal because the first order conditions' for competitive equilibrium and Pareto optimality are not the same.

5.3 Production Externality

We now show that allocation of resources may not be efficient also for the case of externality in production. To show this, consider a simple economy with two firms. Firm 1 produces an output x which will be sold in a competitive market. However, production of x imposes an externality cost denoted by $e(x)$ to firm 2, which is assumed to be convex and strictly increasing.

Let y be the output produced by firm 2, which is sold in competitive market.

Let $c_x(x)$ and $c_y(y)$ be the cost functions of firms 1 and 2 which are both convex and strictly increasing.

The profits of the two firms:

$$\pi_1 = p_x x - c_x(x) \tag{5.16}$$

$$\pi_2 = p_y y - c_y(y) - e(x) \tag{5.17}$$

where p_x and p_y are the prices of x and y, respectively. Then, by the first order conditions, we have for positive amounts of outputs:

$$p_x = c'_x(x) \tag{5.18}$$

$$p_y = c'_y(y) \tag{5.19}$$

However, the profit maximizing output x_c from the first order condition is too large from a social point of view. The first firm only takes account of the private cost – the cost that is imposed on itself– but it ignores the social cost – the private cost plus the cost it imposes on the other firm.

What's the social efficient output?

The social profit, $\pi_1 + \pi_2$ is not maximized at x_c and y_c which satisfy (5.18) and (5.19). If the two firms merged so as to internalize the externality

$$\max_{x,y} \; p_x x + p_y y - c_x(x) - e(x) - c_y(y) \tag{5.20}$$

which gives the first order conditions:

$$p_x = c_x'(x^*) + e'(x^*)$$
$$p_y = c_y'(y^*)$$

where x^* is an efficient amount of output; it is characterized by price being equal to the social cost. Thus, production of x^* is less than the competitive output in the externality case by the convexity of $e(x)$ and $c_x(x)$.

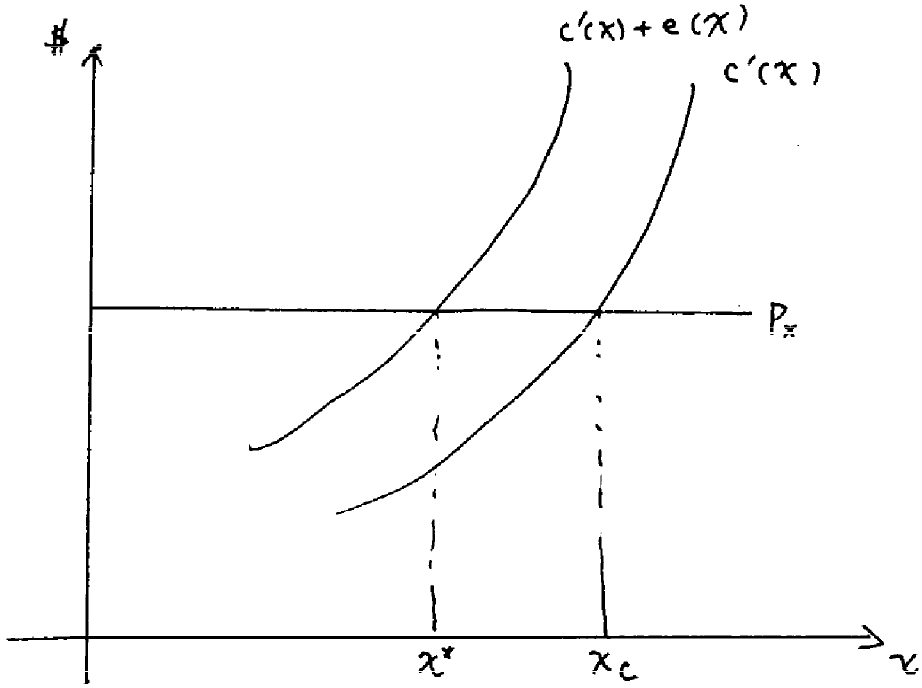

Figure 5.1: The efficient output x^* is less than the competitive output x_c.

5.4 Solutions to Externalities

From the above discussion, we know that a competitive market in general may not result in Pareto efficient outcome, and one needs to seek some other alternative mechanisms to solve the market failure problem. In this section, we now introduce some remedies to this market failure of externality such as:

1. Pigovian taxes
2. Voluntary negotiation (Coase Approach)
3. Compensatory tax/subsidy

4. Creating a missing markets with property rights

5. Direct intervention

6. Merges

7. Incentives mechanism design

Any of the above solution may result in Pareto efficient outcomes, but may lead to different income distributions. Also, it is important to know what kind of information are required to implement a solution listed above.

Most of the above proposed solutions need to make the following assumptions:

1. The source and degree of the externality is identifiable.

2. The recipients of the externality is identifiable.

3. The causal relationship of the externality can be established objectively.

4. The cost of preventing (by different methods) an externality are perfectly known to everyone.

5. The cost of implementing taxes and subsides is negligible.

6. The cost of voluntary negotiation is negligible.

5.4.1 Pigovian Tax

Set a tax rate, t, such that $t = e'(x^*)$. This tax rate to firm 1 would internalize the externality.

$$\pi_1 = p_x \cdot x - c_x(x) - t \cdot x \tag{5.21}$$

The first order condition is:

$$p_x = c_x'(x) + t = c_x'(x) + e'(x^*), \tag{5.22}$$

which is the same as the one for social optimality. That is, when firm 1 faces the wrong price of its action, and a correction tax $t = e'(x^*)$ should be imposed that will lead to a social optimal outcome that is less that of competitive equilibrium outcome. Such correction taxes are called Pigovian taxes.

The problem with this solution is that it requires that the taxing authority knows the externality cost $e(x)$. But, how does the authority know the externality and how do they estimate the value of externality in real world? If the authority knows this information, it might as well just tell the firm how much to produce in the first place. So, in most case, it will not work well.

5.4.2 Coase Voluntary Negotiation

Coase made an observation that in the presence of externalities, the victim has an incentive to pay the firm to stop the production if the victim can compensate the firm by paying $p_x - c_x'(x^*)$.

Remark 5.4.1 Both the Pigovian tax solution and the Coase voluntary negotiation is equivalent in the sense that it achieves Pareto efficient allocation. But, they are different in resulting the income distribution.

Property Rights Responses

To solve the externality problem, Nobel laureate Ronald Coase in a famous article in 1960 argues that government should simply rearrange property rights with appropriately designed property rights. Market then could take care of externalities without direct government intervention.

Example 5.4.1 Two firms: One is chemical factory that discharges chemicals into a river and the other is the fisherman. Suppose the river can produce a value of $50,000. If the chemicals pollute the river, the fish cannot be eaten. How does one solve the externality? Coase's method states that as long as the property rights of the river are clearly assigned, it results in efficient outcomes. That is, the government should give the ownership of the lake either to the chemical firm or to the fisherman, then it will yield an efficient output. To see this, assume that:

The cost of the filter is denoted by c_f.

Case 1: The lake is given to the factory.

i) $c_f < \$50,000$. The fisherman is willing to buy a filter for the factory. The fisherman will pay for the filter so that the chemical cannot pollute the lake.

ii) $c_f > \$50,000$ – The chemical is discharged into the lake. The fisherman does not want to install any filter.

Case 2: The lake is given to the fisherman, and the firm's net product revenue is grater than c_f.

i) $c_f < \$50,000$ – The factory buys the filter so that the chemical cannot pollute the lake.

ii) $c_f > \$50,000$ – The firm pays \$50,000 to the fisherman then the chemical is discharged into the lake.

More generally, let $b(y)$ be the benefit that the polluter draws from a level of pollutant production y and $c(y)$ the cost thus imposed on the pollutee. When b is concave and c increasing and convex, the optimal pollution level is given

$$b'(y^*) = c'(y^*).$$

Suppose that the status quo y_0 corresponds to a situation where $b'(y_0) < c'(y_0)$, and thus the pollution level is too high. Then the polluter and the pollutee have an interest in negotiating. Let ϵ be a small positive number, and assume that the polluter proposes to lower the pollution level to $(y_0 - \epsilon)$ against a payment of $t\epsilon$, where t is comprised between $b'(y_0)$ and $c'(y_0)$. Since $t > b'(y_0)$, this offer raises the polluter's profit; and it is equally beneficial for the pollutee, since $t < c'(y_0)$. Therefore, the two parties will agree to move to a slightly lower pollution level. The reasoning does not stop here: so long as $b'(y_o) < c'(y_0)$, it is possible to lower the pollution level against a well-chosen transfer from pollutee to polluter. The end result is the optimal pollution level. A very similar argument applies in the case where $b'(y_0) > c'(y_0)$.

The formal statement of Coase Theorem thus can be set forth as follows:

Theorem 5.4.1 (Coase Theorem) *When the transaction cost is negligible and there is no income effect, no matter how the property rights are assigned, the resulting outcomes are efficient.*

The problem of this Coase theorem is that, costs of negotiation and organization, in general, are not negligible, and the income effect may not be zero. In fact, Hurwicz (Japan and the World Economy 7, 1995, pp. 49-74) has proved that, even when the transition cost is zero, absence of income effects is not only sufficient (which is well known) but also necessary for Coase Theorem to be true. Thus, a privatization is optimal only in case of zero transaction cost and no income effect.

5.4.3 Missing Market

We can regard externality as a lack of a market for an "externality." For the above example in Pigovian taxes, a missing market is a market for pollution. Adding a market for firm 2 to express its demand for pollution - or for a reduction of pollution - will provide a mechanism for efficient allocations. By adding this market, firm 1 can decide how much pollution it wants to sell, and firm 2 can decide how much pollution it wants to buy.

Let r be the price of pollution.

$x_1 =$ the units of pollution that firm 1 wants to sell;

$x_2 =$ the units of pollution for firm 2 wants to buy.

Normalize the output of firm 1 to x_1.

The profit maximization problems become:

$$\begin{aligned}\pi_1 &= p_x x_1 + r x_1 - c_1(x_1)\\ \pi_2 &= p_y y - r x_2 - e_2(x_2) - c_y(y)\end{aligned}$$

The first order conditions are:

$$p_x + r = c_1'(x_1) \quad \text{for Firm 1}$$

$$p_y = c_y'(y) \quad \text{for Firm 2}$$

$$-r = e'(x_2) \quad \text{for Firm 2.}$$

At the market equilibrium, $x_1^* = x_2^* = x^*$, we have

$$p_x = c_1'(x^*) + e'(x^*) \tag{5.23}$$

which results in a social optimal outcome.

5.4.4 The Compensation Mechanism

The Pigovian taxes were not adequate in general to solve externalities due to the information problem: the tax authority cannot know the cost imposed by the externality. How can one solve this incomplete information problem?

Varian (AER 1994) proposed an incentive mechanism which encourages the firms to correctly reveal the costs they impose on the other. Here, we discuss this mechanism. In brief, a mechanism consists of a message space and an outcome function (rules of game). We will introduce in detail the mechanism design theory in Part III.

Strategy Space (Message Space): $M = M_1 \times M_2$ with $M_1 = \{(t_1, x_1)\}$, where t_1 is interpreted as a Pigovian tax proposed by firm 1 and x_1 is the proposed level of output by firm 1, and t_2 is interpreted as a Pigovian tax proposed by firm 2 and y_2 is the proposed level of output by firm 2.

The mechanism has two stages:

Stage 1: (Announcement stage): Firms 1 and 2 name Pigovian tax rates, t_i, $i = 1, 2$, which may or may not be the efficient level of such a tax rate.

Stage 2: (Choice stage): If firm 1 produces x units of pollution, firm 1 must pay $t_2 x$ to firm 2. Thus, each firm takes the tax rate as given. Firm 2 receives $t_1 x$ units as compensation. Each firm pays a penalty, $(t_1 - t_2)^2$, if they announce different tax rates.

Thus, the payoffs of two firms are:

$$\begin{aligned}
\pi_1^* &= \max_x \; p_x x - c_x(x) - t_2 x - (t_1 - t_2)^2 \\
\pi_2^* &= \max_y \; p_y y - c_y(y) + t_1 x - e(x) - (t_1 - t_2)^2.
\end{aligned}$$

Because this is a two-stage game, we may use the subgame perfect equilibrium, i.e., an equilibrium in which each firm takes into account the repercussions of its first-stage choice on the outcomes in the second stage. As usual, we solve this game by looking at stage 2 first.

At stage 2, firm 1 will choose $x(t_2)$ to satisfy the first order condition:

$$p_x - c_x'(x) - t_2 = 0 \tag{5.24}$$

Note that, by the convexity of c_x, i.e., $c_x''(x) > 0$, we have

$$x'(t_2) = -\frac{1}{c_x''(x)} < 0. \tag{5.25}$$

Firm 2 will choose y to satisfy $p_y = c_y'(y)$.

Stage 1: Each firm will choose the tax rate t_1 and t_2 to maximize their payoffs.

For Firm 1,

$$\max_{t_1} p_x x - c_x(x) - t_2 x(t_2) - (t_1 - t_2)^2 \tag{5.26}$$

which gives us the first order condition:

$$2(t_1 - t_2) = 0$$

so the optimal solution is

$$t_1^* = t_2. \tag{5.27}$$

For Firm 2,

$$\max_{t_2} p_y y - c_y(y) + t_1 x(t_2) - e(x(t_2)) - (t_1 - t_2)^2 \tag{5.28}$$

so that the first order condition is

$$t_1 x'(t_2) - e'(x(t_2))x'(t_2) + 2(t_1 - t_2) = 0$$

and thus

$$[t_1 - e'(x(t_2))]x'(t_2) + 2(t_1 - t_2) = 0. \tag{5.29}$$

By (5.25),(5.27) and (5.29), we have

$$t^* = e'(x(t^*)) \quad \text{with} \qquad t^* = t_1^* = t_2^*. \tag{5.30}$$

Substituting the equilibrium tax rate, $t^* = e'(x(t^*))$, into (**??**) we have

$$p_x = c_x'(x^*) + e'(x^*) \tag{5.31}$$

which is the condition for social efficiency of production.

Remark 5.4.2 This mechanism works by setting opposing incentives for two agents. Firm 1 always has an incentive to match the announcement of firm 2. But consider firm 2's incentive. If firm 2 thinks that firm 1 will propose a large compensation rate t_1 for him, he wants firm 1 to be taxed as little as possible so that firm 1 will produce as much as possible. On the other hand, if firm 2 thinks firm 1 will propose a small t_1, it wants firm 1 to be taxed as much as possible. Thus, the only point where firm 2 is indifferent about the level of production of firm 1 is where firm 2 is exactly compensated for the cost of the externality.

In general, the individual's objective is different from the social goal. However, we may be able to construct an appropriated mechanism so that the individual's profit maximizing goal is consistent with the social goal such as efficient allocations. Tian (2003a) also gave the solution to the consumption externalities by giving the incentive mechanism that results in Pareto efficient allocations. Tian (2003b) study the informational efficiency problem of the mechanisms that results in Pareto efficient allocations for consumption externalities.

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Chapter 6

Public Goods

6.1 Introduction

A public good is a special case of externalities. A *pure public good* is a good in which consuming one unit of the good by an individual in no way prevents others from consuming the same unit of the good. Thus, the good is nonexcludable and non-rival.

A good is *excludable* if people can be excluded from consuming it. A good is *non-rival* if one person's consumption does not reduce the amount available to other consumers.

Examples of Public Goods: street lights, policemen, fire protection, highway system, national defence, flood-control project, public television and radio broadcast, public parks, and a public project.

Local Public Goods: when there is a location restriction for the service of a public good.

Even if the competitive market is an efficient social institution for allocating private goods in an efficient manner, it turns out that a private market is not a very good mechanism for allocating public goods.

6.2 Notations and Basic Settings

In a general setting of public goods economy that includes consumers, producers, private goods, and public goods.

Let

n: the number of consumers.

L: the number of private goods.

K: the number of public goods.

$Z_i \subseteq \Re^L_+ \times \Re^K_+$: the consumption space of consumer i.

$Z \subseteq \Re^{nL}_+ \times \Re^K_+$: consumption space.

$x_i \in \Re^L_+$: a consumption of private goods by consumer i.

$y \in \Re^K_+$: a consumption/production of public goods.

$w_i \in \Re^L_+$: the initial endowment of private goods for consumer i. For simplicity, it is assumed that there is no public goods endowment, but they can be produced from private goods by a firm.

$v \in \Re^L_+$: the private goods input. For simplicity, assume there is only one firm to produce the public goods.

f: $\Re^L_+ \to \Re^K_+$: production function with $y = f(x)$.

θ_i: the profit share of consumer i from the production.

$(x_i, y) \in Z_i$.

$(x, y) = (x_i, ..., x_n, y) \in Z$: an allocation.

$\succcurlyeq_i$ (or u_i if exists) is a preference ordering.

$e_i = (Z_i, \succcurlyeq_i, w_i, \theta_i)$: the characteristic of consumer i.

An allocation $z \equiv (x, y)$ is feasible if

$$\sum_{i=1}^{n} x_i + v \leqq \sum_{i=1}^{n} w_i \tag{6.1}$$

and

$$y = f(v) \tag{6.2}$$

$e = (e_1, ..., e_n, f)$: a public goods economy.

An allocation (x, y) is *Pareto efficient* for a public goods economy e if it is feasible and there is no other feasible allocation (x', y') such that $(x'_i, y') \succcurlyeq_i (x_i, y)$ for all consumers i and $(x'_k, y') \succ_k (x_k, y)$ for some k.

An allocation (x, y) is *weakly Pareto efficient* for the public goods economy e if it is feasible and there is no other feasible allocation (x', y') such that $(x'_i, y') \succ_i (x_i, y)$ for all consumers i.

Remark 6.2.1 Unlike private goods economies, even though under the assumptions of continuity and strict monotonicity, a weakly Pareto efficient allocation may not be Pareto efficient for the public goods economies. The following proposition is due to Tian (Economics Letters, 1988).

Proposition 6.2.1 *For the public goods economies, a weakly Pareto efficient allocation may not be Pareto efficient even if preferences satisfy strict monotonicity and continuity.*

Proof: The proof is by way of an example. Consider an economy with $(n, L, K) = (3, 1, 1)$, constant returns in producing y from x (the input-output coefficient normalized to one), and the following endowments and utility functions: $w_1 = w_2 = w_3 = 1$, $u_1(x_1, y) = x_1 + y$, and $u_i(x_i, y) = x_i + 2y$ for $i = 2, 3$. Then $z = (x, y)$ with $x = (0.5, 0, 0)$ and $y = 2.5$ is weakly Pareto efficient but not Pareto efficient because $z' = (x', y') = (0, 0, 0, 3)$ Pareto-dominates z by consumers 2 and 3.

However, under an additional condition of strict convexity, they are equivalent. The proof is left to readers.

6.3 Discrete Public Goods

6.3.1 Efficient Provision of Public Goods

For simplicity, consider a public good economy with n consumers and two goods: one private good and one public good.

Let g_i be the contribution made by consumer i, so that

$$\begin{aligned} x_i + g_i &= w_i \\ \sum_{i=1}^{n} g_i &= v \end{aligned}$$

Assume $u_i(x_i, y)$ is strictly monotonic increasing and continuous.

Let c be the cost of producing the public project so that the production technology is given by

$$y = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} g_i \geqq c \\ 0 & \text{otherwise} \end{cases}.$$

We first want to know under what conditions providing the public good will be Pareto dominate to not producing it, i.e., there exists $(g_1, \ldots, g_n)$ such that $\sum_{i=1}^{n} g_i \geqq c$ and

$$u_i(w_i - g_i, 1) > u_i(w_i, 0) \quad \forall i.$$

Let r_i be the maximum willingness-to-pay (reservation price) of consumer i, i.e., r_i must satisfy

$$u_i(w_i - r_i, 1) = u_i(w_i, 0). \tag{6.3}$$

If producing the public project Pareto dominates not producing the public project, we have

$$u_i(w_i - g_i, 1) > u_i(w_i, 0) = u_i(w_i - r_i, 1) \text{ for all } i \tag{6.4}$$

By monotonicity of u_i, we have

$$w_i - g_i > w_i - r_i \quad \text{for } i \tag{6.5}$$

Then, we have

$$r_i > g_i \tag{6.6}$$

and thus

$$\sum_{i=1}^{n} r_i > \sum_{i=1}^{n} g_i \geqq c \tag{6.7}$$

That is, the sum of the willingness-to-pay for the public good must excess the cost of providing it. This condition is necessary. In fact, this condition is also sufficient. In summary, we have the following proposition.

Proposition 6.3.1 *Providing the public good Pareto dominates not producing the good if and only if* $\sum_{i=1}^{n} r_i > \sum_{i=1}^{n} g_i \geqq c$.

6.3.2 Free-Rider Problem

How effective is a private market at providing public goods? The answer as shown below is that we cannot expect that purely independent decision will necessarily result in an efficient amount of the public good being produced. To see this, suppose

$$
\begin{aligned}
r_i &= 100 \quad i = 1, 2 \\
c &= 150 \text{ (total cost)} \\
g_i &= \begin{cases} 150/2 = 75 & \text{if both agents make contributions} \\ 150 & \text{if only agent } i \text{ makes contribution} \end{cases}
\end{aligned}
$$

Each person decides independently whether or not to buy the public good. As a result, each one has an incentive to be a free-rider on the other as shown the following payoff matrix.

	buy	*doesn't buy*
buy	(25, 25)	(-50, 100)
doesn't buy	(100, -50)	(0, 0)

Note that net payoffs are defined by $r_i - g_i$. Thus, it is given by 100 - 150/2 = 25 when both consumers are willing to produce the public project, and 100-150 = -50 when only one person wants to buy, but the other person does not.

This is the prisoner's dilemma. The dominant strategy equilibrium in this game is (doesn't buy, doesn't buy). Thus, no body wants to share the cost of producing the public project, but wants to free-ride on the other consumer. As a result, the public good is not provided at all even thought it would be efficient to do so. Thus, voluntary contribution in general does not result in the efficient level of the public good.

6.3.3 Voting for a Discrete Public Good

The amount of a public good is often determined by a voting. Will this generally results in an efficient provision? The answer is no.

Voting does not result in efficient provision. Consider the following example.

Example 6.3.1

$$
\begin{aligned}
c &= 99 \\
r_1 &= 90, \; r_2 = 30, \; r_3 = 30
\end{aligned}
$$

Clearly, $r_1 + r_2 + r_3 > c$. $g_i = 99/3 = 33$. So the efficient provision of the public good should be yes. However, under the majority rule, only consumer 1 votes "yes" since she receives a positive net benefit if the good is provided. The 2nd and 3rd persons vote "no" to produce public good, and therefore, the public good will not be provided so that we have inefficient provision of the public good. The problem with the majority rule is that it only measures the net benefit for the public good, whereas the efficient condition requires a comparison of willingness-to-pay.

6.4 Continuous Public Goods

6.4.1 Efficient Provision of Public Goods

Again, for simplicity, we assume there is only one public good and one private good that may be regarded as money, and $y = f(v)$.

The welfare maximization approach shows that Pareto efficient allocations can be characterized by

$$\begin{aligned} &\max_{(x,y)} \sum_{i=1}^{n} a_i u_i(x_i, y) \\ s.t. \quad &\sum_{i=1}^{n} x_i + v \leqq \sum_{i=1}^{n} w_i \\ &y \leqq f(v) \end{aligned}$$

Define the Lagrangian function:

$$L = \sum_{i=1}^{n} a_i u_i(x_i, y) + \lambda \left(\sum_{i=1}^{n} w_i - \sum_{i=1}^{n} x_i - v \right) + \mu(f(v) - y). \tag{6.8}$$

When u_i is strictly quasi-concave and differentiable and $f(v)$ is concave and differentiable, the set of interior Pareto optimal allocations are characterized by the first order condition:

$$\frac{\partial L}{\partial x_i} = 0: \qquad a_i \frac{\partial u_i}{\partial x_i} = \lambda \tag{6.9}$$

$$\frac{\partial L}{\partial v} = 0: \qquad \mu f'(v) = \lambda \tag{6.10}$$

$$\frac{\partial L}{\partial y} = 0: \qquad \sum_{i=1}^{n} a_i \frac{\partial u_i}{\partial y} = \mu. \tag{6.11}$$

By (6.9) and (6.10)

$$\frac{a_i}{\mu} = \frac{f'(v)}{\frac{\partial u_i}{\partial x_i}} \tag{6.12}$$

Substituting (6.12) into (6.11),

$$\sum_{i=1}^{n} \frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = \frac{1}{f'(v)}. \tag{6.13}$$

Thus, we obtain the well-known Lindahl-Samuelson condition.

In conclusion, the conditions for Pareto efficiency are given by

$$\begin{cases} \sum_{i=1}^{n} MRS_{yx_i}^{i} = MRTS_{yv} \\ \sum x_i + v \leqq \sum_{i=1}^{n} w_i \\ y = f(v) \end{cases} \tag{6.14}$$

Example 6.4.1

$$\begin{aligned} u_i &= a_i \ln y + \ln x_i \\ y &= v \end{aligned}$$

the Lindahl-Samuelson condition is

$$\sum_{i=1}^{n} \frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = 1 \tag{6.15}$$

and thus

$$\sum_{i=1}^{n} \frac{\frac{a_i}{y}}{\frac{1}{x_i}} = \sum_{i=1}^{n} \frac{a_i x_i}{y} = 1 \Rightarrow \sum a_i x_i = y \tag{6.16}$$

which implies the level of the public good is not uniquely determined.

Thus, in general, the marginal willingness-to-pay for a public good depends on the amount of private goods consumption, and therefor, the efficient level of y depends on x_i. However, in the case of quasi-linear utility functions,

$$u_i(x_i, y) = x_i + u_i(y) \tag{6.17}$$

the Lindahl-Samuelson condition becomes

$$\sum_{i=1}^{n} u_i'(y) = \frac{1}{f'(v)} \equiv c'(y) \tag{6.18}$$

and thus y is uniquely determined.

Example 6.4.2

$$\begin{aligned} u_i &= a_i \ln y + x_i \\ y &= v \end{aligned}$$

the Lindahl-Samuelson condition is

$$\sum_{i=1}^{n} \frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = 1 \tag{6.19}$$

and thus

$$\sum_{i=1}^{n} \frac{a_i}{y} = \sum_{i=1}^{n} \frac{a_i}{y} = 1 \Rightarrow \sum a_i = y \tag{6.20}$$

which implies the level of the public good is uniquely determined.

6.4.2 Lindahl Equilibrium

We have given the conditions for Pareto efficiency in the presence of public goods. The next problem is how to achieve a Pareto efficient allocation in a decentralized way. In private-goods-only economies, any competitive equilibrium is Pareto optimal. However, with public goods, a competitive mechanism does not help. For instance, if we tried the competitive solution with a public good and two consumers, the utility maximization would equalize the MRS and the relative price, e.g.,

$$MRS^A_{yx} = MRS^B_{yx} = \frac{p_y}{p_x}.$$

This is an immediate violation to the Samuelson-Lindahl optimal condition.

Lindahl suggested to use a tax method to provide a public good. Each person is signed a specific "personalized price" for the public good. The Lindahl solution is a way to mimic the competitive solution in the presence of public goods. Suppose we devise a mechanism to allocate the production cost of a public good between consumers, then we can achieve the Samuelson-Lindahl condition. To this end, we apply different prices of a public good to different consumers. The idea of the Lindahl solution is that the consumption level of a public goods is the same to all consumers, but the price of the public good is personalized among consumers in the way that the price ratio of two goods for each person being equal the marginal rates of substitutions of two goods.

To see this, consider a public goods economy e with $x_i \in R^L_+$ (private goods) and $y \in R^K_+$ (public goods). For simplicity, we assume the CRS for $y = f(v)$. A feasible allocation

$$\sum_{i=1}^{n} x_i + v \leqq \sum_{i=1}^{n} w_i \tag{6.21}$$

Let $q_i \in R^K_+$ be the personalized price vector of consumer i for consuming the public goods.

Let $q = \sum_{i=1}^{n} q_i$: the market price vector of y.

Let $p \in R^L_+$ be the price vector of private goods.

The profit is defined as $\pi = qy - pv$ with $y = f(v)$.

Remark 6.4.1 Because of CRS, the maximum profit is zero at the Lindahl equilibrium.

Definition 6.4.1 (Lindahl Equilibrium) An allocation $(x^*, y^*) \in R^{nL+K}_+$ is a Lindahl equilibrium allocation if it is feasible and there exists a price vector $p^* \in K^L_+$ and personalized price vectors $q^*_i \in R^K_+$, one for each individual i, such that

(i) $p^* x^*_i + q^*_i y^* \leqq p w_i$;

(ii) $(x_i, y) \succ_i (x^*_i, y^*)$ implies $p^* x_i + q^*_i y > p^* x^*_i + q^*_i y^*$;

(iii) $q^* y^* - p^*_i v^* = 0$,

where $v^* = \sum_{t=1}^{n} w_i - \sum_{i=1}^{n} x^*_i$ and $\sum_{t=1}^{n} q^*_i = q^*$.

We call $(x^*, y^*, p^*, q^*_1, \ldots, q^*_n)$ a Lindahl equilibrium.

We may regard a Walrasian equilibrium as a special case of a Lindahl equilibrium when there are no public goods. Similarly, we have the following First Fundamental Theorem of Welfare Economics for public goods economies.

Theorem 6.4.1 *: Every Lindahl allocation (x^*, y^*) with the price system $(p^*, q^*_i, \ldots, q^*_n)$ is weakly Pareto efficient, and further under local non-satiation, it is Pareto efficient.*

Proof: We only prove the second part. The first part is the simple. Suppose not. There exists another feasible allocation (x_i, y) such that $(x_i, y) \succcurlyeq_i (x^*_i, y^*)$ for all i and $(x_j, y) \succ_j (x^*_j, y^*)$ for some j. Then, by local-non-satiation of $\succcurlyeq_i$, we have

$$\begin{aligned} p^* x_i + q^*_i y &\geqq p^* w_i \quad \text{for all } i = 1, 2, \ldots, n \\ p^* x_j + q^*_j y &> p^* w_j \quad \text{for some } j. \end{aligned}$$

Thus

$$\sum_{i=1}^{n} p^* x_i + \sum_{i=1}^{n} q_i y > \sum_{i=1}^{n} p^* w_i \tag{6.22}$$

So

$$p^* \sum_{i=1}^{n} x_i + q^* y > \sum_{i=1}^{n} p^* w_i$$

or

$$p^* \sum_{i=1}^{n} x_i + pv > \sum_{i=1}^{n} p^* w_i$$

by noting that $q^*y - p^*v \leqq q^*y^* - p^*v^* = 0$. Hence,

$$p^* \left[\sum_{i=1}^{n} (x_i - w_i) + v \right] > 0$$

which contradicts the fact that (x, y) is feasible.

For a public economy with one private good and one public good $y = \frac{1}{q}v$, the definition of Lindahl equilibrium becomes much simpler.

An allocation (x^*, y^*) is a Lindahl Allocation if (x^*, y^*) is feasible (i.e., $\sum_{i=1}^{n} x_i^* + qy^* \leqq \sum w_i$) and there exists q_i^*, $i = 1, \ldots, n$ such that

(i) $x_i^* + q_i y^* \leqq w_i$

(ii) $(x_i, y) \succ_i (x_i^*, y^*)$ implies $x_i + q_i y > w_i$

(iii) $\sum_{i=1}^{n} q_i = q$

In fact, the feasibility condition is automatically satisfied when the budget constraints (i) is satisfied.

If (x^*, y^*) is an interior Lindahl equilibrium allocation, from the utility maximization, we can have the first order condition:

$$\frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial y}} = \frac{q_i}{1} \tag{6.23}$$

which means the Lindahl-Samuelson condition holds:

$$\sum_{i=1}^{n} MRS_{yx_i} = q,$$

which is the necessary condition for Pareto efficiency.

Example 6.4.3

$$
\begin{aligned}
u_i(x_i, y) &= x_i^{\alpha_i} y^{(1-\alpha_i)} \quad for \;\; 0 < \alpha_i < 1 \\
y &= \frac{1}{q} v
\end{aligned}
$$

The budget constraint is:

$$x_i + q_i y = w_i.$$

The demand functions for x_i and y_i of each i are given by

$$x_i = \alpha_i w_i \tag{6.24}$$

$$y_i = \frac{(1-\alpha_i)w_i}{q_i} \tag{6.25}$$

Since $y_1 = y_2 = \ldots y_n = y^*$ at the equilibrium, we have by (6.25)

$$q_i y^* = (1-\alpha_i)w_i. \tag{6.26}$$

Making summation, we have

$$q y^* = \sum_{i=1}^{n}(1-\alpha_i)w_i.$$

Then, we have

$$y^* = \frac{\sum_{i=1}^{n}(1-\alpha_i)w_i}{q}$$

and thus, by (6.26), we have

$$q_i = \frac{(1-\alpha_i)w_i}{y^*} = \frac{q(1-\alpha_i)w_i}{\sum_{i=1}^{n}(1-\alpha_i)w_i}. \tag{6.27}$$

If we want to find a Lindahl equilibrium, we must know the preferences or MRS of each consumer.

But because of the free-rider problem, it is very difficult for consumers to report their preferences truthfully.

6.4.3 Free-Rider Problem

When the MRS is known, a Pareto efficient allocation (x, y) can be determined from the Lindahl-Samuelson condition or the Lindahl solution. After that, the contribution of each consumer is given by $g_i = w_i - x_i$. However, the society is hard to know the information about MRS. Of course, a naive method is that we could ask each individual to reveal his preferences, and thus determine the willingness-to-pay. However, since each consumer is self-interested, each person wants to be a free-rider and thus is not willing to tell the true MRS. If consumers realize that shares of the contribution for producing public goods (or the personalized prices) depend on their answers, they have "incentives to cheat." That is, when the consumers are asked to report their utility functions or MRSs, they will have incentives to report a smaller MRS so that they can pay less, and consume the public good (free riders). This causes the major difficulty in the public economies.

To see this, notice that the social goal is to reach Pareto efficient allocations for the public goods economy, but from the personal interest, each person solves the following problem:

$$\max u_i(x_i, y) \tag{6.28}$$

subject to

$$\begin{aligned} g_i &\in (0, w_i) \\ x_i + g_i &= w_i \\ y &= f(g_i + \sum_{j \neq i}^{n} g_j). \end{aligned}$$

That is, each consumer i takes others' strategies g_{-i} as given, and maximizes his payoffs. From this problem, we can form a non-cooperative game:

$$\Gamma = (G_i, \phi_i)_{i=1}^n$$

where $G_i = [0, w_i]$ is the strategy space of consumer i and $\phi_i : G_1 \times G_2 \times ... \times G_n \to R$ is the payoff function of i which is defined by

$$\phi_i(g_i, g_{-i}) = u_i[(w_i - g_i), f(g_i + \sum_{j \neq i}^{n} g_i)] \tag{6.29}$$

Definition 6.4.2 For the game, $\Gamma = (G_i, \phi_i)_{i=1}^n$, the strategy $g^* = (g_1^*, ..., g_n^*)$ is a *Nash Equilibrium* if

$$\phi_i(g_i^*, g_{-i}^*) \geqq \phi_i(g_i, g_{-i}^*) \quad \text{for all } g_i \in G_i \text{ and all } i = 1, 2, ..., n,$$

g^* is a *dominant strategy equilibrium* if

$$\phi_i(g_i^*, g_{-i}) \geqq \phi_i(g_i, g_{-i}) \quad \text{for all } g \in G \text{ and all } i = 1, 2, ...$$

Remark 6.4.2 Note that the difference between Nash equilibrium (NE) and dominant strategy is that at NE, given best strategy of others, each consumer chooses his best strategy while dominant strategy means that the strategy chosen by each consumer is best regardless of others' strategies. Thus, a dominant strategy equilibrium is clearly a Nash equilibrium, but the converse may not be true. Only for a very special payoff functions, there is a dominant strategy while a Nash equilibrium exists for a continuous and quasi-concave payoff functions that are defined on a compact set.

For Nash equilibrium, if u_i and f are differentiable, then an interior solution g must satisfy the first order condition:

$$\frac{\partial \phi_i(g^*)}{\partial g_i} = 0 \quad \text{for all } i = 1, \dots, n. \tag{6.30}$$

Thus, we have

$$\frac{\partial \phi_i}{\partial g_i} = \frac{\partial u_i}{\partial x_i}(-1) + \frac{\partial u_i}{\partial y} f'(g_i^* + \sum_{j \neq i}^{n} g_j) = 0$$

So,

$$\frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = \frac{1}{f'(g_i^* + \sum_{j \neq i} g_j)},$$

and thus

$$MRS^i_{yx_i} = MRTS_{yv},$$

which does not satisfy the Lindahl-Samuelson condition. Thus, the Nash equilibrium in general does not result in Pareto efficient allocations. The above equation implies that the low level of public good is produced rather than the Pareto efficient level of the public good. Therefore, Nash equilibrium allocations are in general not consistent with Pareto efficient allocations. How can one solve this free-ride problem? We will answer this question in the mechanism design theory.

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Part III

Incentives, Information, and Mechanism Design

The notion of incentives is a basic and key concept in modern economics. To many economists, economics is to a large extent a matter of incentives: incentives to work hard, to produce good quality products, to study, to invest, to save, etc.

Until about 30 year ago, economics was mostly concerned with understanding the theory of value in large economies. A central question asked in general equilibrium theory was whether a certain mechanism (especially the competitive mechanism) generated Pareto-efficient allocations, and if so – for what categories of economic environments. In a perfectly competitive market, the pressure of competitive markets solves the problem of incentives for consumers and producers. The major project of understanding how prices are formed in competitive markets can proceed without worrying about incentives.

The question was then reversed in the economics literature: instead of regarding mechanisms as given and seeking the class of environments for which they work, one seeks mechanisms which will implement some desirable outcomes (especially those which result in Pareto-efficient and individually rational allocations) for a given class of environments without destroying participants' incentives, and which have a low cost of operation and other desirable properties. In a sense, the theorists went back to basics.

The reverse question was stimulated by two major lines in the history of economics. Within the capitalist/private-ownership economics literature, a stimulus arose from studies focusing upon the failure of the competitive market to function as a mechanism for implementing efficient allocations in many nonclassical economic environments such as the presence of externalities, public goods, incomplete information, imperfect competition, increasing return to scale, etc. At the beginning of the seventies, works by Akerlof (1970), Hurwicz (1972), Spence (1974), and Rothschild and Stiglitz (1976) showed in various ways that asymmetric information was posing a much greater challenge and could not be satisfactorily imbedded in a proper generalization of the Arrow-Debreu theory.

A second stimulus arose from the socialist/state-ownership economics literature, as evidenced in the "socialist controversy" — the debate between Mises-Hayek and Lange-Lerner in twenties and thirties of the last century. The controversy was provoked by von Mises's skepticism as to even a theoretical feasibility of rational allocation under socialism.

The incentives structure and information structure are thus two basic features of any economic system. The study of these two features is attributed to these two major lines, culminating in the theory of mechanism design. The theory of economic mechanism design which was originated by Hurwicz is very general. All economic mechanisms and systems (including those known and unknown, private-ownership, state-ownership, and mixed-ownership systems) can be studied with this theory.

At the micro level, the development of the theory of incentives has also been a major advance in economics in the last thirty years. Before, by treating the firm as a black box the theory remains silent on how the owners of firms succeed in aligning the objectives of its various members, such as workers, supervisors, and managers, with profit maximization.

When economists began to look more carefully at the firm, either in agricultural or managerial economics, incentives became the central focus of their analysis. Indeed, delegation of a task to an agent who has different objectives than the principal who delegates this task is problematic when information about the agent is imperfect. This problem is the essence of incentive questions. Thus, **conflicting objectives and decentralized information are the two basic ingredients of incentive theory**.

We will discover that, in general, these informational problems prevent society from achieving the first-best allocation of resources that could be possible in a world where all information would be common knowledge. The additional costs that must be incurred because of the strategic behavior of privately informed economic agents can be viewed as one category of the transaction costs. Although they do not exhaust all possible transaction costs, economists have been rather successful during the last thirty years in modelling and analyzing these types of costs and providing a good understanding of the limits set by these on the allocation of resources. This line of research also provides a whole set of insights on how to begin to take into account agents' responses to the incentives provided by institutions.

We will briefly present the incentive theory in three chapters. Chapters 7 and 8 consider the principal-agent model where the principal delegates an action to a single agent with private information. This private information can be of two types: either the agent can take an action unobserved by the principal, the case of moral hazard or hidden action; or the agent has some private knowledge about his cost or valuation that is ignored by the principal, the case of adverse selection or hidden knowledge. Incentive theory considers when this private information is a problem for the principal, and what is the optimal way for the principal to cope with it. **The design of the principal's optimal contract can be regarded as a simple optimization problem. This simple focus will turn out to be enough to highlight the various trade-offs between allocative efficiency and distribution of information rents arising under incomplete information**. The mere existence of informational constraints may generally prevent the principal from achieving allocative efficiency. We will characterize the allocative distortions that the principal finds desirable to implement in order to mitigate the impact of informational constraints.

Chapter 9 will consider situations with one principal and many agents. Asymmetric information may not only affect the relationship between the principal and each of his agents, but it may also plague the relationships between agents. Moreover, maintaining the hypothesis that agents adopt an individualistic behavior, those organizational contexts require a solution concept of equilibrium, which describes the strategic interaction between agents under complete or incomplete information.

Chapter 7

Principal-Agent Model: Hidden Information

7.1 Introduction

Incentive problems arise when a principal wants to delegate a task to an agent with private information. The exact opportunity cost of this task, the precise technology used, and how good the matching is between the agent's intrinsic ability and this technology are all examples of pieces of information that may become private knowledge of the agent. In such cases, we will say that there is adverse selection.

Eexample

1. The landlord delegates the cultivation of his land to a tenant, who will be the only one to observe the exact local weather conditions.

2. A client delegates his defense to an attorney who will be the only one to know the difficulty of the case.

3. An investor delegates the management of his portfolio to a broker, who will privately know the prospects of the possible investments.

4. A stockholder delegates the firm's day-to-day decisions to a manager, who will be the only one to know the business conditions.

5. An insurance company provides insurance to agents who privately know how good a driver they are.

6. The Department of Defense procures a good from the military industry without

knowing its exact cost structure.

7. A regulatory agency contracts for service with a Public Utility without having complete information about its technology.

The common aspect of all those contracting settings is that the information gap between the principal and the agent has some fundamental implications for the design of the contract they sign. In order to reach an efficient use of economic resources, some information rent must be given up to the privately informed agent. At the optimal second-best contract, the principal trades off his desire to reach allocative efficiency against the costly information rent given up to the agent to induce information revelation. Implicit here is the idea that there exists a legal framework for this contractual relationship. The contract can be enforced by a benevolent court of law, the agent is bounded by the terms of the contract.

The main objective of this chapter is to characterize the optimal rent extraction-efficiency trade-off faced by the principal when designing his contractual offer to the agent under the set of incentive feasible constraints: incentive and participation constraints. In general, incentive constraints are binding at the optimum, showing that adverse selection clearly impedes the efficiency of trade. **The main lessons of this optimization is that the optimal second-best contract calls for a distortion in the volume of trade away from the first-best and for giving up some strictly positive information rents to the most efficient agents**.

7.2 The Basic Model

7.2.1 Economic Environment (Technology, Preferences, and Information)

Consider a consumer or a firm (the principal) who wants to delegate to an agent the production of q units of a good. The value for the principal of these q units is $S(q)$ where $S' > 0$, $S'' < 0$ and $S(0) = 0$.

The production cost of the agent is unobservable to the principal, but it is common knowledge that the fixed cost is F and the marginal cost belongs to the set $\Phi = \{\underline{\theta}, \bar{\theta}\}$. The agent can be either efficient ($\underline{\theta}$) or inefficient ($\bar{\theta}$) with respective probabilities ν and

$1 - \nu$. That is, he has the cost function

$$C(q, \underline{\theta}) = \underline{\theta} q + F \quad \text{with probability } \nu \tag{7.1}$$

or

$$C(q, \bar{\theta}) = \bar{\theta} q + F \quad \text{with probability } 1 - \nu \tag{7.2}$$

Denote by $\Delta\theta = \bar{\theta} - \underline{\theta} > 0$.

7.2.2 Contracting Variables: Outcomes

The contracting variables are the quantity produced q and the transfer t received by the agent. Let $\mathcal{A}$ be the set of feasible allocations that is given by

$$\mathcal{A} = \{(q, t) : q \in \Re_+, t \in \Re\} \tag{7.3}$$

These variables are both observable and verifiable by a third party such as a benevolent court of law.

7.2.3 Timing

Unless explicitly stated, we will maintain the timing defined in the figure below, where A denotes the agent and P the principal.

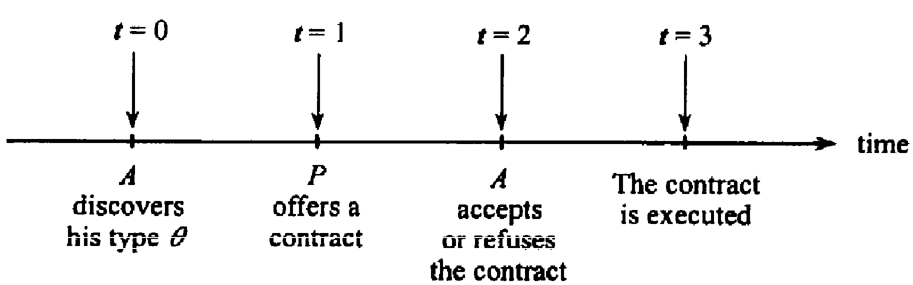

Figure 7.1: Timing of contracting under hidden information.

Note that contracts are offered at the interim stage; there is already asymmetric information between the contracting parties when the principal makes his offer.

7.3 The Complete Information Optimal Contract(Benchmark Case)

7.3.1 First-Best Production Levels

To get a reference system for comparison, let us first suppose that there is no asymmetry of information between the principal and the agent. The efficient production levels are obtained by equating the principal's marginal value and the agent's marginal cost. Hence, we have the following first-order conditions

$$S'(\underline{q}^*) = \underline{\theta} \tag{7.4}$$

and

$$S'(\bar{q}^*) = \bar{\theta}. \tag{7.5}$$

The complete information efficient production levels $\underline{q}^*$ and $\bar{q}^*$ should be both carried out if their social values, respectively $\underline{W}^* = S(\underline{q}^*) - \underline{\theta}\underline{q}^* - F$, and $\overline{W}^* = S(\bar{q}^*) - \bar{\theta}\bar{q}^* - F$, are non-negative.

Since

$$S(\underline{q}^*) - \underline{\theta}\underline{q}^* \geqq S(\bar{q}^*) - \underline{\theta}\bar{q}^* \geqq S(\bar{q}^*) - \bar{\theta}\bar{q}^*$$

by definition of $\underline{\theta}$ and $\bar{\theta} > \underline{\theta}$, the social value of production when the agent is efficient, $\underline{W}^*$, is greater than when he is inefficient, namely $\overline{W}^*$.

For trade to be always carried out, it is thus enough that production be socially valuable for the least efficient type, i.e., the following condition must be satisfied

$$\bar{W}^* = S(\bar{q}^*) - \bar{\theta}\bar{q}^* - F \geqq 0. \tag{7.6}$$

As the fixed cost F plays no role other than justifying the existence of a single agent, it is set to zero from now on in order to simplify notations.

Note that, since the principal's marginal value of output is decreasing, the optimal production of an efficient agent is greater than that of an inefficient agent, i.e., $\underline{q}^* > \bar{q}^*$.

7.3.2 Implementation of the First-Best

For a successful delegation of the task, the principal must offer the agent a utility level that is at least as high as the utility level that the agent obtains outside the relationship.

We refer to these constraints as the agent's participation constraints. If we normalize to zero the agent's outside opportunity utility level (sometimes called his quo utility level), these participation constraints are written as

$$\underline{t} - \underline{\theta}\underline{q} \geqq 0, \tag{7.7}$$

$$\bar{t} - \bar{\theta}\bar{q} \geqq 0. \tag{7.8}$$

To implement the first-best production levels, the principal can make the following take-it-or-leave-it offers to the agent: If $\theta = \bar{\theta}$ (resp. $\underline{\theta}$), the principal offers the transfer $\bar{t}^*$ (resp. $\underline{t}^*$) for the production level $\bar{q}^*$(resp. $\underline{q}^*$) with $\bar{t}^* = \bar{\theta}\bar{q}^*$(resp.$\underline{t}^* = \underline{\theta}\underline{q}^*$). Thus, whatever his type, the agent accepts the offer and makes zero profit. The complete information optimal contracts are thus $(\underline{t}^*, \underline{q}^*)$ if $\theta = \underline{\theta}$ and $(\bar{t}^*, \bar{q}^*)$ if $\theta = \bar{\theta}$. Importantly, under complete information delegation is costless for the principal, who achieves the same utility level that he would get if he was carrying out the task himself (with the same cost function as the agent).

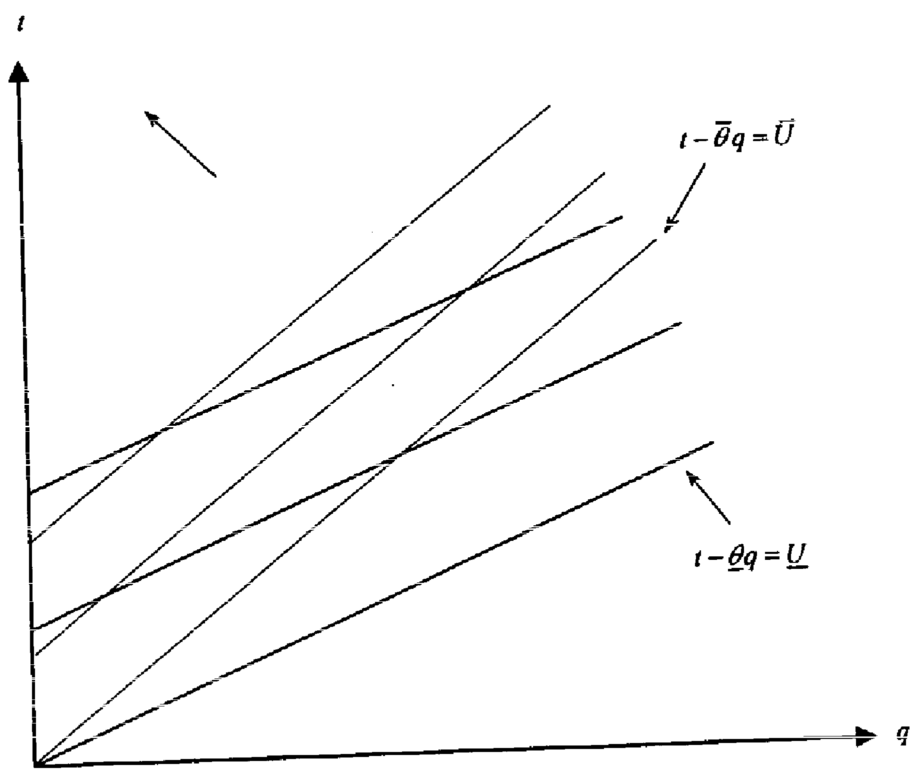

Figure 7.2: Indifference curves of both types.

7.3.3 A Graphical Representation of the Complete Information Optimal Contract

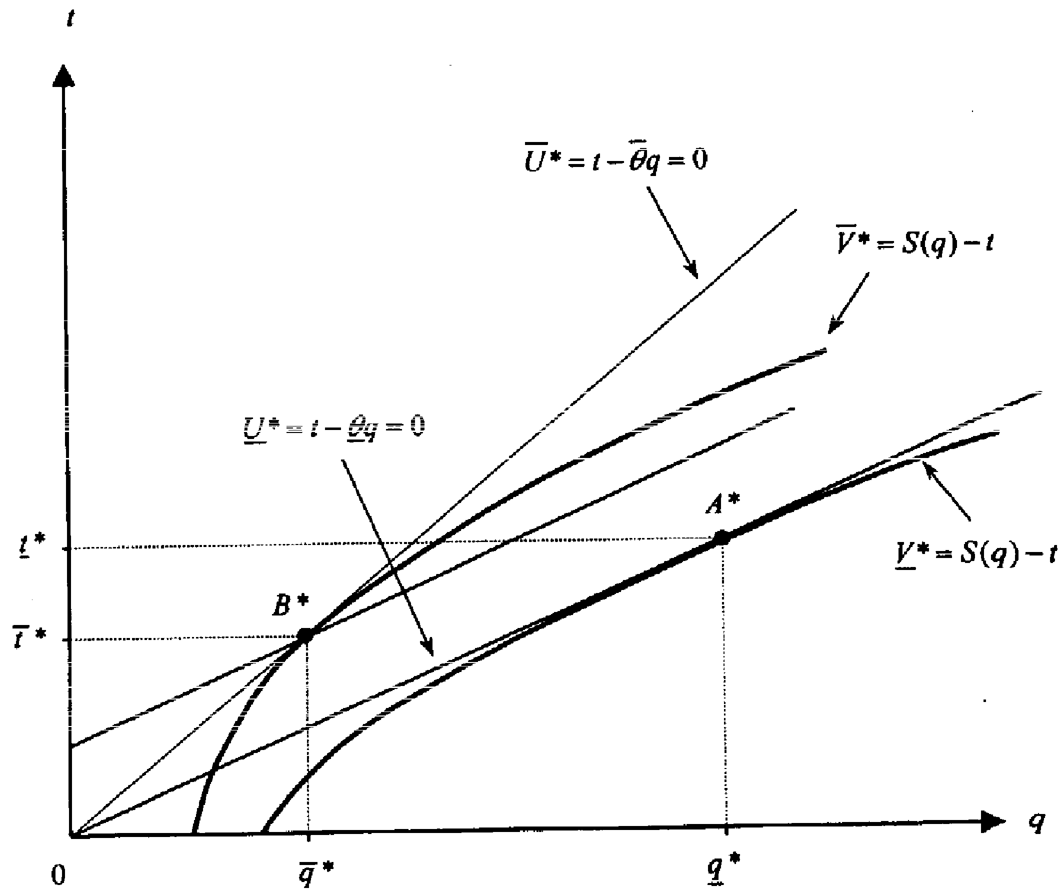

Figure 7.3: First best contracts.

Since $\bar{\theta} > \underline{\theta}$, the iso-utility curves for different types cross only once as shown in the above figure. This important property is called the single-crossing or Spence-Mirrlees property.

The complete information optimal contract is finally represented Figure 7.3 by the pair of points (A^*, B^*). Note that since the iso-utility curves of the principal correspond to increasing levels of utility when one moves in the southeast direction, the principal reaches a higher profit when dealing with the efficient type. We denote by $\bar{V}^*$ (resp. $\underline{V}^*$) the principal's level of utility when he faces the $\bar{\theta}-$ (resp. $\underline{\theta}-$) type. Because the principal's has all the bargaining power in designing the contract, we have $\bar{V}^* = W^*$ (resp. $\underline{V}^* = \underline{W}^*$) under complete information.

7.4 Incentive Feasible Contracts

7.4.1 Incentive Compatibility and Participation

Suppose now that the marginal cost θ is the agent's private information and let us consider the case where the principal offers the menu of contracts $\{(\underline{t}^*, \underline{q}^*); (\bar{t}^*, \bar{q}^*)\}$ hoping that an agent with type $\underline{\theta}$ will select $(\underline{t}^*, \underline{q}^*)$ and an agent with $\bar{\theta}$ will select instead $(\bar{t}^*, \bar{q}^*)$.

From Figure 7.3 above, we see that B^* is preferred to A* by both types of agents. Offering the menu (A^*, B^*) fails to have the agents self-selecting properly within this menu. The efficient type have incentives to mimic the inefficient one and selects also contract B^*. The complete information optimal contracts can no longer be implemented under asymmetric information. We will thus say that the menu of contracts $\{(\underline{t}^*, \underline{q}^*); (\bar{t}^*, \bar{q}^*)\}$ is not incentive compatible.

Definition 7.4.1 *A menu of contracts $\{(\underline{t}, \underline{q}); (\bar{t}, \bar{q})\}$ is incentive compatible when $(\underline{t}, \underline{q})$ is weakly preferred to $(\bar{t}, \bar{q})$ by agent $\underline{\theta}$ and $(\bar{t}, \bar{q})$ is weakly preferred to $(\underline{t}, \underline{q})$ by agent $\bar{\theta}$.*

Mathematically, these requirements amount to the fact that the allocations must satisfy the following *incentive compatibility constraints*:

$$\underline{t} - \underline{\theta}\underline{q} \geqq \bar{t} - \underline{\theta}\bar{q} \tag{7.9}$$

and

$$\bar{t} - \bar{\theta}\bar{q} \geqq \underline{t} - \bar{\theta}\underline{q} \tag{7.10}$$

Furthermore, for a menu to be accepted, it must satisfy the following two *participation constraints*:

$$\underline{t} - \underline{\theta}\underline{q} \geqq 0, \tag{7.11}$$

$$\bar{t} - \bar{\theta}\bar{q} \geqq 0. \tag{7.12}$$

Definition 7.4.2 *A menu of contracts is incentive feasible if it satisfies both incentive and participation constraints (7.9) through (7.12).*

The inequalities (7.9) through (7.12) express additional constraints imposed on the allocation of resources by asymmetric information between the principal and the agent.

7.4.2 Special Cases

Bunching or Pooling Contracts: A first special case of incentive feasible menu of contracts is obtained when the contracts targeted for each type coincide, i.e., when $\underline{t} = \bar{t} = t^p, \underline{q} = \bar{q} = q^p$ and both types of agent accept this contract.

Shutdown of the Least Efficient Type: Another particular case occurs when one of the contracts is the null contract (0,0) and the nonzero contract (t^s, q^s) is only accepted by the efficient type. Then, (7.9) and (7.11) both reduce to

$$t^s - \underline{\theta} q^s \geqq 0. \tag{7.13}$$

The incentive constraint of the bad type reduces to

$$0 \geqq t^s - \bar{\theta} q^s. \tag{7.14}$$

7.4.3 Monotonicity Constraints

Incentive compatibility constraints reduce the set of feasible allocations. Moreover, these quantities must generally satisfy a monotonicity constraint which does not exist under complete information. Adding (7.9) and (7.10), we immediately have

$$\underline{q} \geqq \bar{q}. \tag{7.15}$$

We will call condition (7.15) an implementability condition that is necessary and sufficient for implementability.

7.5 Information Rents

To understand the structure of the optimal contract it is useful to introduce the concept of information rent.

We know from previous discussion, under complete information, the principal is able to maintain all types of agents at their zero status quo utility level. Their respective utility levels $\underline{U}^*$ and $\bar{U}^*$ at the first-best satisfy

$$\underline{U}^* = \underline{t}^* - \underline{\theta} \underline{q}^* = 0 \tag{7.16}$$

and

$$\bar{U}^* = \bar{t}^* - \bar{\theta}\bar{q}^* = 0. \tag{7.17}$$

Generally this will not be possible anymore under incomplete information, at least when the principal wants both types of agents to be active.

Take any menu $\{(\bar{t}, \bar{q}); (\underline{t}, \underline{q})\}$ of incentive feasible contracts and consider the utility level that a $\underline{\theta}$-agent would get by mimicking a $\bar{\theta}$-agent. The high-efficient agent would get

$$\bar{t} - \underline{\theta}\bar{q} = \bar{t} - \bar{\theta}\bar{q} + \Delta\theta\bar{q} = \bar{U} + \Delta\theta\bar{q}. \tag{7.18}$$

Thus, as long as the principal insists on a positive output for the inefficient type, $\bar{q} > 0$, the principal must give up a positive rent to a $\underline{\theta}$-agent. This information rent is generated by the informational advantage of the agent over the principal.

We use the notations $\underline{U} = \underline{t} - \underline{\theta}\underline{q}$ and $\bar{U} = \bar{t} - \bar{\theta}\bar{q}$ to denote the respective information rent of each type.

7.6 The Optimization Program of the Principal

According to the timing of the contractual game, the principal must offer a menu of contracts before knowing which type of agent he is facing. Then, the principal's problem writes as

$$\max_{\{(\bar{t},\bar{q});(\underline{t},\underline{q})\}} \nu(S(\underline{q}) - \underline{t}) + (1-\nu)(S(\bar{q}) - \bar{t})$$
$$\text{subject to (7.9) to (7.12).}$$

Using the definition of the information rents $\underline{U} = \underline{t} - \underline{\theta}\underline{q}$ and $\bar{U} = \bar{t} - \bar{\theta}\bar{q}$, we can replace transfers in the principal's objective function as functions of information rents and outputs so that the new optimization variables are now $\{(\underline{U}, \underline{q}); (\bar{U}, \bar{q})\}$. The focus on information rents enables us to assess the distributive impact of asymmetric information, and the focus on outputs allows us to analyze its impact on allocative efficiency and the overall gains from trade. Thus an allocation corresponds to a volume of trade and a distribution of the gains from trade between the principal and the agent.

With this change of variables, the principal's objective function can then be rewritten as

$$\underbrace{\nu(S(\underline{q}) - \underline{\theta}\underline{q}) + (1-\nu)(S(\bar{q}) - \bar{\theta}\bar{q})} - \underbrace{(\nu\underline{U} + (1-\nu)\bar{U})}. \tag{7.19}$$

The first term denotes expected allocative efficiency, and the second term denotes expected information rent which implies that the principal is ready to accept some distortions away from efficiency in order to decrease the agent's information rent.

The incentive constraints (7.9) and (7.10), written in terms of information rents and outputs, becomes respectively

$$\underline{U} \geqq \bar{U} + \Delta\theta\bar{q}, \tag{7.20}$$

$$\bar{U} \geqq \underline{U} - \Delta\theta\underline{q}. \tag{7.21}$$

The participation constraints (7.11) and (7.12) become respectively

$$\underline{U} \geqq 0, \tag{7.22}$$

$$\bar{U} \geqq 0. \tag{7.23}$$

The principal wishes to solve problem (P) below:

$$\max_{\{(\underline{U},\underline{q});(\bar{U},\bar{q})\}} \nu(S(\underline{q}) - \underline{\theta}\underline{q}) + (1-\nu)(S(\bar{q}) - \bar{\theta}\bar{q}) - (\nu\underline{U} + (1-\nu)\bar{U})$$

subject to (7.20) to (7.23).

We index the solution to this problem with a superscript SB, meaning second-best.

7.7 The Rent Extraction-Efficiency Trade-Off

7.7.1 The Optimal Contract Under Asymmetric Information

The major technical difficulty of problem (P) is to determine which of the many constraints imposed by incentive compatibility and participation are the relevant ones. i.e., the binding ones at the optimum or the principal's problem.

Let us first consider contracts without shutdown, i.e., such that $\bar{q} > 0$. This is true when the so-called Inada condition $S'(0) = +\infty$ is satisfied and $\lim_{q\to 0} S'(q)q = 0$.

Note that the $\underline{\theta}$-agent's participation constraint (7.22) is always strictly-satisfied. Indeed, (7.23) and (7.20) immediately imply (7.22). (7.21) also seems irrelevant because the difficulty comes from a $\underline{\theta}$-agent willing to claim that he is inefficient rather than the reverse.

This simplification in the number of relevant constraints leaves us with only two remaining constraints, the $\underline{\theta}$-agent's incentive constraint (7.20) and the $\bar{\theta}$-agent's participation constraint (7.23), and both constraints must be binding at the optimum of the principal's problem (P):

$$\underline{U} = \Delta\theta\bar{q} \tag{7.24}$$

and

$$\bar{U} = 0. \tag{7.25}$$

Substituting (7.24) and (7.25) into the principal's objective function, we obtain a reduced program (P') with outputs as the only choice variables:

$$\max_{\{(\underline{q},\bar{q})\}} \nu(S(\underline{q}) - \underline{\theta}\underline{q}) + (1-\nu)(S(\bar{q}) - \bar{\theta}\bar{q}) - (\nu\Delta\theta\bar{q}).$$

Compared with the full information setting, asymmetric information alters the principal's optimization simply by the subtraction of the expected rent that has to be given up to the efficient type. The inefficient type gets no rent, but the efficient type $\underline{\theta}$ gets information rent that he could obtain by mimicking the inefficient type θ. This rent depends only on the level of production requested from this inefficient type.

The first order conditions are then given by

$$S'(\underline{q}^{SB}) = \underline{\theta} \quad \text{or } \underline{q}^{SB} = \underline{q}^*. \tag{7.26}$$

and

$$(1-\nu)(S'(\bar{q}^{SB}) - \bar{\theta}) = \nu\Delta\theta. \tag{7.27}$$

(7.27) expresses the important trade-off between efficiency and rent extraction which arises under asymmetric information.

To validate our approach based on the sole consideration of the efficient type's incentive constraint, it is necessary to check that the omitted incentive constraint of an inefficient agent is satisfied. i.e., $0 \geqq \Delta\theta\bar{q}^{SB} - \Delta\theta\underline{q}^{SB}$. This latter inequality follows from the monotonicity of the second-best schedule of outputs since we have $\underline{q}^{SB} = \underline{q}^* > \bar{q}^* > \bar{q}^{SB}$.

In summary, we have the following proposition.

Proposition 7.7.1 *Under asymmetric information, the optimal menu of contracts entails:*

(1) No output distortion for the efficient type test in respect to the first-best, $\underline{q}^{SB} = \underline{q}^*$. A downward output distortion for the inefficient type, $\bar{q}^{SB} < \bar{q}^*$ with

$$S'(\bar{q}^{SB}) = \bar{\theta} + \frac{\nu}{1-\nu}\Delta\theta. \tag{7.28}$$

(2) Only the efficient type gets a positive information rent given by

$$\underline{U}^{SB} = \Delta\theta\bar{q}^{SB}. \tag{7.29}$$

(3) The second-best transfers are respectively given by $\underline{t}^{SB} = \underline{\theta}\underline{q}^* + \Delta\theta\bar{q}^{SB}$ and $\bar{t}^{SB} = \bar{\theta}\bar{q}^{SB}$.

7.7.2 A Graphical Representation of the Second-Best Outcome

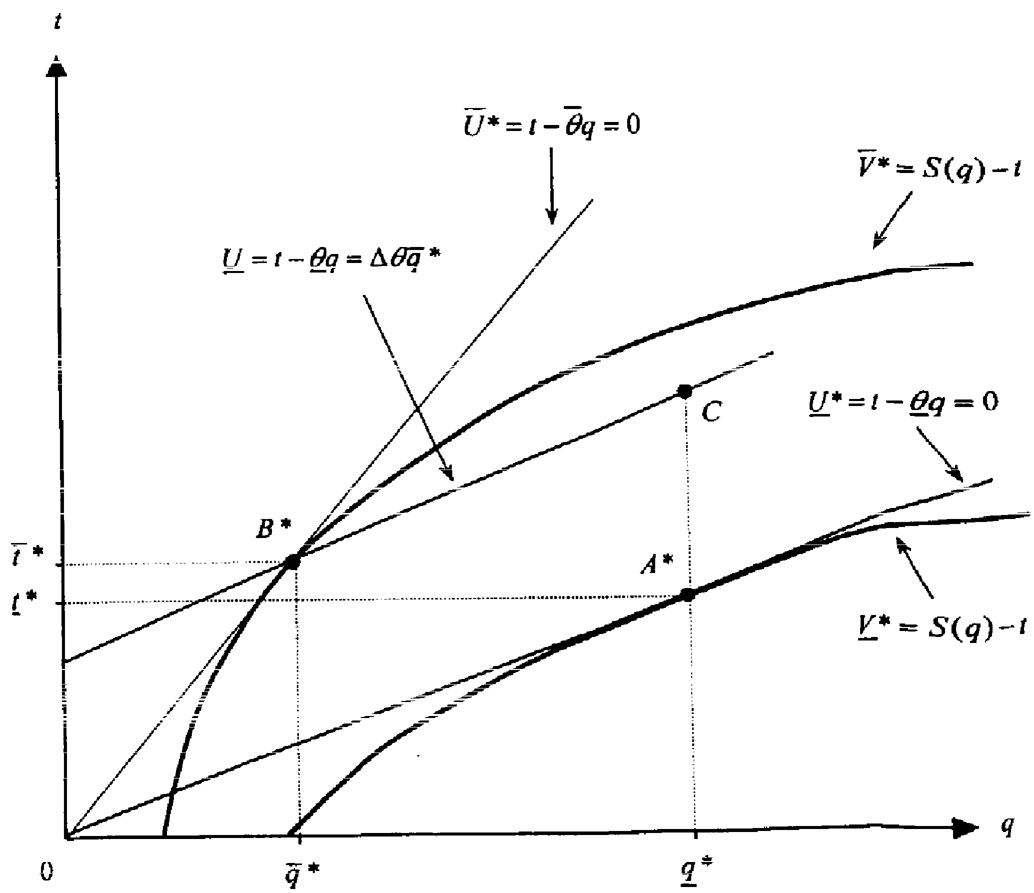

Figure 7.4: Rent needed to implement the first best outputs.

Starting from the complete information optimal contract (A^*, B^*) that is not incentive compatible, we can construct an incentive compatible contract (B^*, C) with the same production levels by giving a higher transfer to the agent producing $\underline{q}^*$ as shown in the figure above. The contract C is on the $\underline{\theta}$- agent's indifference curve passing through B^*. Hence, the $\underline{\theta}$-agent is now indifferent between B^* and C. (B^*, C) becomes an incentive-compatible menu of contracts. The rent that is given up to the $\underline{\theta}$-firm is now $\Delta\theta\bar{q}^*$. This contract is not optimal by the first order conditions (7.26) and (7.27). The optimal trade-off finally occurs at (A^{SB}, B^{SB}) as shown in the figure below.

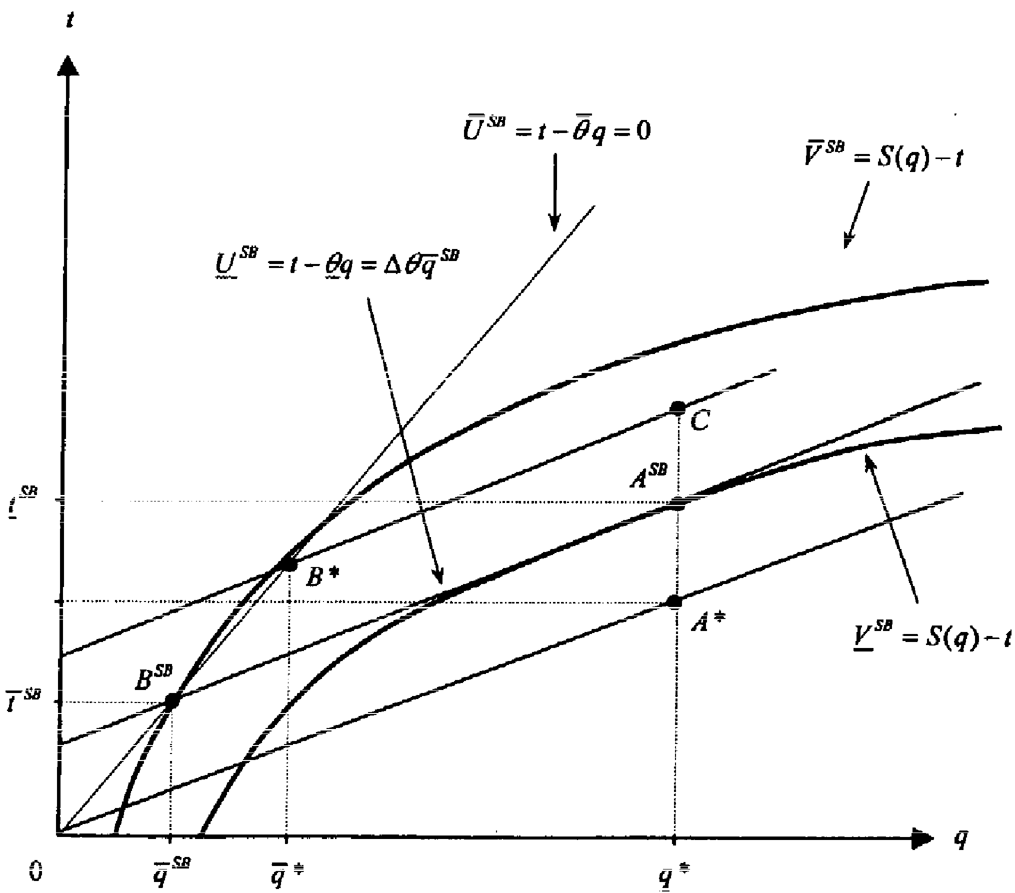

Figure 7.5: Optimal second-best contract S^{SB} and B^{SB}.

7.7.3 Shutdown Policy

If the first-order condition in (7.28) has no positive solution, $\bar{q}^{SB}$ should be set at zero. We are in the special case of a contract with shutdown. B^{SB} coincides with 0 and A^{SB} with A^* in the figure above. No rent is given up to the $\underline{\theta}$-firm by the unique non-null contract $(\underline{t}^*, \underline{q}^*)$ offered and selected only by agent $\underline{\theta}$. The benefit of such a policy is that no rent is given up to the efficient type.

Remark 7.7.1 The shutdown policy is dependent on the status quo utility levels. Suppose that, for both types, the status quo utility level is $U_0 > 0$. Then, from the principal's objective function, we have

$$\frac{\nu}{1-\nu}\Delta\theta\bar{q}^{SB} + U_0 \geqq S(\bar{q}^{SB}) - \bar{\theta}\bar{q}^{SB}. \tag{7.30}$$

Thus, for ν large enough, shutdown occurs even if the Inada condition $S'(0) = +\infty$ is satisfied. Note that this case also occurs when the agent has a strictly positive fixed cost $F > 0$ (to see that, just set $U_0 = F$).

7.8 The Theory of the Firm Under Asymmetric Information

When the delegation of task occurs within the firm, a major conclusion of the above analysis is that, because of asymmetric information, the firm does not maximize the social value of trade, or more precisely its profit, a maintained assumption of most economic theory. This lack of allocative efficiency should not be considered as a failure in the rational use of resources within the firm. Indeed, the point is that allocative efficiency is only one part of the principal's objective. The allocation of resources within the firm remains constrained optimal once informational constraints are fully taken into account.

Williamson (1975) has advanced the view that various transaction costs may impede the achievement of economic transactions. Among the many origins of these costs, Williamson stresses informational impact as an important source of inefficiency. Even in a world with a costless enforcement of contracts, a major source of allocative inefficiency is the existence of asymmetric information between trading partners.

Even though asymmetric information generates allocative inefficiencies, those efficiencies do not call for any public policy motivated by reasons of pure efficiency. Indeed, any benevolent policymaker in charge of correcting these inefficiencies would face the same informational constraints as the principal. The allocation obtained above is Pareto optimal in the set of incentive feasible allocations or incentive Pareto optimal.

7.9 Asymmetric Information and Marginal Cost Pricing

Under complete information, the first-best rules can be interpreted as price equal to marginal cost since consumers on the market will equate their marginal utility of consumption to price.

Under asymmetric information, price equates marginal cost only when the producing firm is efficient ($\theta = \bar{\theta}$). Using (7.28), we get the expression of the price $p(\bar{\theta})$ for the inefficient types output

$$p(\bar{\theta}) = \bar{\theta} + \frac{\nu}{1-\nu}\Delta\theta. \tag{7.31}$$

Price is higher than marginal cost in order to decrease the quantity $\bar{q}$ produced by the inefficient firm and reduce the efficient firm's information rent. Alternatively, we can say that price is equal to a generalized (or virtual) marginal cost that includes, in addition to the traditional marginal cost of the inefficient type $\bar{\theta}$, an information cost that is worth $\frac{\nu}{1-\nu}\Delta\theta$.

7.10 The Revelation Principle

In the above analysis, we have restricted the principal to offer a menu of contracts, one for each possible type. One may wonder if a better outcome could be achieved with a more complex contract allowing the agent possibly to choose among more options. The revelation principle ensures that there is no loss of generality in restricting the principal to offer simple menus having at most as many options as the cardinality of the type space. Those simple menus are actually examples of direct revelation mechanisms.

Definition 7.10.1 *A direct revelation mechanism is a mapping $g(\cdot)$ from Θ to $\mathcal{A}$ which writes as $g(\theta) = (q(\theta), t(\theta))$ for all belonging to Θ. The principal commits to offer the transfer $t(\tilde{\theta})$ and the production level $q(\tilde{\theta})$ if the agent announces the value $\tilde{\theta}$ for any $\tilde{\theta}$ belonging to Θ.*

Definition 7.10.2 *A direct revelation mechanism $g(\cdot)$ is truthful if it is incentive compatible for the agent to announce his true type for any type, i.e., if the direct revelation* *mechanism satisfies the following incentive compatibility constraints:*

$$t(\underline{\theta}) - \underline{\theta}q(\underline{\theta}) \geqq t(\bar{\theta}) - \underline{\theta}q(\bar{\theta}), \tag{7.32}$$

$$t(\bar{\theta}) - \bar{\theta}q(\bar{\theta}) \geqq t(\underline{\theta} - \bar{\theta}q(\bar{\theta}). \tag{7.33}$$

Denoting transfer and output for each possible report respectively as $t(\underline{\theta}) = \underline{t}, q(\underline{\theta}) = \underline{q}$, $t(\bar{\theta}) = \bar{t}$ and $q(\bar{\theta}) = \bar{q}$, we get back to the notations of the previous sections.

A more general mechanism can be obtained when communication between the principal and the agent is more complex than simply having the agent report his type to the principal.

Let M be the message space offered to the agent by a more general mechanism.

Definition 7.10.3 *A mechanism is a message space M and a mapping $\tilde{q}(\cdot)$ from M to $\mathcal{A}$ which writes as $\tilde{g}(m) = (\tilde{q}(m), \tilde{t}(m))$ for all m belonging to M.*

When facing such a mechanism, the agent with type θ chooses a best message $m^*(\theta)$ that is implicitly defined as

$$\tilde{t}(m^*(\theta)) - \theta\tilde{q}(m^*(\theta)) \geqq \tilde{t}(\tilde{m}) - \theta\tilde{q}(\tilde{m}) \quad \text{for all } \tilde{m} \in M. \tag{7.34}$$

The mechanism $(M, \tilde{g}(\cdot))$ induces therefore an allocation rule $a(\theta) = (\tilde{q}(m^*(\theta)), \tilde{t}(m^*(\theta)))$ mapping the set of types Θ into the set of allocations $\mathcal{A}$.

Then we have the following revelation principle in the one agent case.

Proposition 7.10.1 *Any allocation rule $a(\theta)$ obtained with a mechanism $(M, \tilde{g}(\cdot))$ can also be implemented with a truthful direct revelation mechanism.*

Proof. The indirect mechanism $(M, \tilde{g}(\cdot))$ induces an allocation rule $a(\theta) = (\tilde{q}(m^*(\theta)),$ $\tilde{t}(m^*(\theta)))$ from into $\mathcal{A}$. By composition of $\tilde{q}(\cdot)$ and $m^*(\cdot)$, we can construct a direct revelation mechanism $g(\cdot)$ mapping Θ into $\mathcal{A}$, namely $g = \tilde{g} \circ m^*$, or more precisely $g(\theta) = (q(\theta), t(\theta)) \equiv \tilde{g}(m^*(\theta)) = (\tilde{q}(m^*(\theta)), \tilde{t}(m^*(\theta)))$ for all $\theta \in \Theta$.

We check now that the direct revelation mechanism $g(\cdot)$ is truthful. Indeed, since (7.34) is true for all $\tilde{m}$, it holds in particular for $\tilde{m} = m^*(\theta')$ for all $\theta' \in \Theta$. Thus we have

$$\tilde{t}(m^*(\theta)) - \theta\tilde{q}(m^*(\theta)) \geqq \tilde{t}(m^*(\theta')) - \theta\tilde{q}(m^*(\theta')) \quad \text{for all } (\theta, \theta') \in \Theta^2. \tag{7.35}$$

Finally, using the definition of $g(\cdot)$, we get

$$t(\theta) - \theta q(\theta) \geqq t(\theta') - \theta q(\theta') \quad \text{for all } (\theta, \theta') \in \Theta^2. \tag{7.36}$$

Hence, the direct revelation mechanism $g(\cdot)$ is truthful.

Importantly, the revelation principle provides a considerable simplification of contract theory. It enables us to restrict the analysis to a simple aid well-defined family of functions, the truthful direct revelation mechanism.

7.11 A More General Utility Function for the Agent

Still keeping quasi-linear utility functions, let $U = t - C(q, \theta)$ now be the agent's objective function in the assumptions: $C_q > 0$, $C_\theta > 0$, $C_{qq} > 0$ and $C_{q\theta} > 0$. The generalization of the Spence- Mirrlees property is now $C_{q\theta} > 0$. This latter condition still ensures that the different types of the agent have indifference curves which cross each other at most once. This Spence-Mirrlees property is quite clear: a more efficient type is also more efficient at the margin.

Incentive feasible allocations satisfy the following incentive and participation constraints:

$$\underline{U} = \underline{t} - C(\underline{q}, \underline{\theta}) \geqq \bar{t} - C(\bar{q}, \underline{\theta}), \tag{7.37}$$

$$\bar{U} = \bar{t} - C(\bar{q}, \bar{\theta}) \geqq \underline{t} - C(\underline{q}, \bar{\theta}), \tag{7.38}$$

$$\underline{U} = \underline{t} - C(\underline{q}, \underline{\theta}) \geqq 0, \tag{7.39}$$

$$\bar{U} = \bar{t} - C(\bar{q}, \bar{\theta}) \geqq 0. \tag{7.40}$$

7.11.1 The Optimal Contract

Just as before, the incentive constraint of an efficient type in (7.37) and the participation constraint of an inefficient type in (7.40) are the two relevant constraints for optimization. These constraints rewrite respectively as

$$\underline{U} \geqq \bar{U} + \Phi(\bar{q}) \tag{7.41}$$

where $\Phi(\bar{q}) = C(\bar{q}, \bar{\theta}) - C(\bar{q}, \underline{\theta})$ (with $\Phi' > 0$ and $\Phi'' > 0$), and

$$\bar{U} \geqq 0. \tag{7.42}$$

Those constraints are both binding at the second-best optimum, which leads to the following expression of the efficient type's rent

$$\underline{U} = \Phi(\bar{q}). \tag{7.43}$$

Since $\Phi' > 0$, reducing the inefficient agent's output also reduces the efficient agent's information rent.

With the assumptions made on $C(\cdot)$, one can also check that the principal's objective function is strictly concave with respect to outputs.

The solution of the principal's program can be summarized as follows:

Proposition 7.11.1 *With general preferences satisfying the Spence-Mirrlees property, $C_{q\theta} > 0$, the optimal menu of contracts entails:*

(1) No output distortion with respect to the first-best outcome for the efficient type, $\underline{q}^{SB} = \underline{q}^*$ with

$$S'(\underline{q}^*) = C_q(\underline{q}^*, \underline{\theta}). \tag{7.44}$$

A downward output distortion for the inefficient type, $\bar{q}^{SB} < \bar{q}^*$ with

$$S'(\bar{q}^*) = C_q(\bar{q}^*, \bar{\theta}) \tag{7.45}$$

and

$$S'(\bar{q}^{SB}) = C_q(\bar{q}^{SB}, \bar{\theta}) + \frac{\nu}{1-\nu}\Phi'(\bar{q}^{SB}). \tag{7.46}$$

(2) Only the efficient type gets a positive information rent given by $\underline{U}^{SB} = \Phi(\bar{q}^{SB})$.

(3) The second-best transfers are respectively given by $\underline{t}^{SB} = C(\underline{q}^*, \underline{\theta}) + \Phi(\bar{q}^{SB})$ and $\bar{t}^{SB} = C(\bar{q}^{SB}, \bar{\theta})$.

The first-order conditions (7.44) and (7.46) characterize the optimal solution if the neglected incentive constraint (7.38) is satisfied. For this to be true, we need to have

$$\begin{aligned} \bar{t}^{SB} - C(\bar{q}^{SB}, \bar{\theta}) &\geqq \underline{t}^{SB} - C(\underline{q}^{SB}, \bar{\theta}), \\ &= \bar{t}^{SB} - C(\bar{q}^{SB}, \underline{\theta}) + C(\underline{q}^{SB}, \underline{\theta}) - C(\underline{q}^{SB}, \bar{\theta}) \end{aligned} \tag{7.47}$$

by noting that (7.37) holds with equality at the optimal output such that $\underline{t}^{SB} = \bar{t}^{SB} - C(\bar{q}^{SB}, \underline{\theta}) + C(\underline{q}^{SB}, \underline{\theta})$. Thus, we need to have

$$0 \geqq \Phi(\bar{q}^{SB}) - \Phi(\underline{q}^{SB}). \tag{7.48}$$

Since $\Phi' > 0$ from the Spence-Mirrlees property, then (7.48) is equivalent to $\bar{q}^{SB} \leqq \underline{q}^{SB}$. But from our assumptions we easily derive that $\underline{q}^{SB} = \underline{q}^* > \bar{q}^* > \bar{q}^{SB}$. So the Spence-Mirrlees property guarantees that only the efficient type's incentive constraint has to be taken into account.

7.11.2 More than Two Goods

Let us now assume that the agent is producing a whole vector of goods $q = (q_1, ..., q_n)$ for the principal. The agents' cost function becomes $C(q, \theta)$ with $C(\cdot)$ being strictly convex in q. The value for the principal of consuming this whole bundle is now $S(q)$ with $S(\cdot)$ being strictly concave in q.

In this multi-output incentive problem, the principal is interested in a whole set of activities carried out simultaneously by the agent. It is straightforward to check that the efficient agent's information rent is now written as $\underline{U} = \Phi(q)$ with $\Phi(q) = C(q, \bar{\theta}) - C(q, \underline{\theta})$. This leads to second- best optimal outputs. The efficient type produces the first-best vector of outputs $\underline{q}^{SB} = \underline{q}^*$ with

$$S_{q_i}(\underline{q}^*) = C_{q_i}(\underline{q}^*, \underline{\theta}) \quad \text{for all } i \in \{1, ..., n\}. \tag{7.49}$$

The inefficient types vector of outputs $\bar{q}^{SB}$ is instead characterized by the first-order conditions

$$S_{q_i}(\bar{q}^{SB}) = C_{q_i}(\bar{q}^{SB}, \bar{\theta}) + \frac{\nu}{1-\nu}\Phi_{q_i}(\bar{q}^{SB}) \quad \text{for all } i \in \{1, ..., n\}, \tag{7.50}$$

which generalizes the distortion of models with a single good.

Without further specifying the value and cost functions, the second-best outputs define a vector of outputs with some components $\bar{q}_i^{SB}$ above $\bar{q}_i^*$ for a subset of indices i.

Turning to incentive compatibility, summing the incentive constraints $\underline{U} \geqq \bar{U} + \Phi(\bar{q})$

and $\bar{U} \geqq \underline{U} - \Phi(\underline{q})$ for any incentive feasible contract yields

$$\begin{aligned} \Phi(\underline{q}) &= C(\underline{q}, \bar{\theta}) - C(\underline{q}, \underline{\theta}) && (7.51) \\ &\geqq C(\bar{q}, \bar{\theta}) - C(\bar{q}, \underline{\theta}) \\ &= \Phi(\bar{q}) \quad \text{for all implementable pairs } (\bar{q}, \underline{q}). && (7.52) \end{aligned}$$

Obviously, this condition is satisfied if the Spence-Mirrlees property $C_{q_i\theta} > 0$ holds for each output i and if the monotonicity conditions $\bar{q}_i < \underline{q}_i$ for all i are satisfied.

7.12 Ex Ante versus Ex Post Participation Constraints

The case of contracts we consider so far is offered at the interim stage, i.e., the agent already knows his type. However, sometimes the principal and the agent can contract at the ex ante stage, i.e., before the agent discovers his type. For instance, the contours of the firm may be designed before the agent receives any piece of information on his productivity. In this section, we characterize the optimal contract for this alternative timing under various assumptions about the risk aversion of the two players.

7.12.1 Risk Neutrality

Suppose that the principal and the agent meet and contract ex ante. If the agent is risk neutral, his ex ante participation constraint is now written as

$$\nu \underline{U} + (1 - \nu)\bar{U} \geqq 0. \tag{7.53}$$

This ex ante participation constraint replaces the two interim participation constraints.

Since the principal's objective function is decreasing in the agent's expected information rent, the principal wants to impose a zero expected rent to the agent and have (7.53) be binding. Moreover, the principal must structure the rents $\underline{U}$ and $\bar{U}$ to ensure that the two incentive constraints remain satisfied. An example of such a rent distribution that is both incentive compatible and satisfies the ex ante participation constraint with an equality is

$$\underline{U}^* = (1 - \nu)\theta\bar{q}^* > 0 \quad \text{and } \bar{U}^* = -\nu\theta\bar{q}^* < 0. \tag{7.54}$$

With such a rent distribution, the optimal contract implements the first-best outputs without cost from the principal's point of view as long as the first-best is monotonic as requested by the implementability condition. In the contract defined by (7.54), the agent is rewarded when he is efficient and punished when he turns out to be inefficient. In summary, we have

Proposition 7.12.1 *When the agent is risk neutral and contracting takes place ex ante, the optimal incentive contract implements the first-best outcome.*

Remark 7.12.1 The principal has in fact much more leeway in structuring the rents $\underline{U}$ and $\bar{U}$ in such a way that the incentive constraints hold and the ex ante participation constraint (7.53) holds with an equality. Consider the following contracts $\{(\underline{t}^*, \underline{q}^*); (\bar{t}^*, \bar{q}^*)\}$ where $\underline{t}^* = S(\underline{q}^*) - T^*$ and $\bar{t}^* = S(\bar{q}^*) - T^*$, with T^* being a lump-sum payment to be defined below. This contract is incentive compatible since

$$\underline{t}^* - \underline{\theta}\underline{q}^* = S(\underline{q}^*) - \underline{\theta}\underline{q}^* - T^* > \bar{t}^* - \underline{\theta}\bar{q}^* = S(\bar{q}^*) - \underline{\theta}\bar{q}^* - T^* \tag{7.55}$$

by definition of q^*, and

$$\bar{t}^* - \bar{\theta}\bar{q}^* = S(\bar{q}^*) - \bar{\theta}\bar{q}^* - T^* > \underline{t}^* - \bar{\theta}\underline{q}^* = S(\underline{q}^*) - \bar{\theta}\underline{q}^* - T^* \tag{7.56}$$

by definition of $\bar{q}^*$.

Note that the incentive compatibility constraints are now strict inequalities. Moreover, the fixed-fee T^* can be used to satisfy the agent's ex ante participation constraint with an equality by choosing $T^* = \nu(S(\underline{q}^*) - \underline{\theta}\underline{q}^*) + (1-\nu)(S(\bar{q}^*) - \bar{\theta}\bar{q}^*)$. This implementation of the first-best outcome amounts to having the principal selling the benefit of the relationship to the risk-neutral agent for a fixed up-front payment T^*. The agent benefits from the full value of the good and trades off the value of any production against its cost just as if he was an efficiency maximizer. We will say that the agent is residual claimant for the firms profit.

7.12.2 Risk Aversion

A Risk-Averse Agent

The previous section has shown us that the implementation of the first-best is feasible with risk neutrality. What happens if the agent is risk-averse?

Consider now a risk-averse agent with a Von Neumann-Morgenstern utility function $u(\cdot)$ defined on his monetary gains $t-\theta q$, such that $u'>0$, $u''<0$ and $u(0)=0$. Again, the contract between the principal and the agent is signed before the agent discovers his type. The incentive constraints are unchanged but the agent's ex ante participation constraint is now written as

$$\nu u(\underline{U})+(1-\nu)u(\bar{U})\geqq 0. \tag{7.57}$$

As usual, one can check (7.21) is slack at the optimum, and thus the principal's program reduces now to

$$\max_{\{(\bar{U},\bar{q});(\underline{U},\underline{q})\}} \nu(S(\underline{q})-\underline{\theta}\underline{q}-\underline{U})+(1-\nu)(S(\bar{q})-\bar{\theta}\bar{q}-\bar{U}),$$

subject to (7.20) and (7.57).

We have the following proposition.

Proposition 7.12.2 *When the agent is risk-averse and contracting takes place ex ante, the optimal menu of contracts entails:*

(1) No output distortion for the efficient $\underline{q}^{SB}=\underline{q}^*$. *A downward output distortion for the inefficient type* $\bar{q}^{SB}<\bar{q}^*$, *with*

$$S'(\bar{q}^{SB})=\bar{\theta}+\frac{\nu(u'(\bar{U}^{SB})-u'(\underline{U}^{SB}))}{\nu u'(\underline{U}^{SB})+(1-\nu)u'(\bar{U}^{SB})}\Delta\theta. \tag{7.58}$$

(2) Both (7.20) and (7.57) are the only binding constraints. The efficient (resp. inefficient) type gets a strictly positive (resp. negative) ex post information rent, $U^{SB}>0>\bar{U}^{SB}$.

Proof: Define the following Lagrangian for the principals problem

$$\begin{aligned} L(\underline{q},\bar{q},\underline{U},\bar{U},\lambda,\mu) &= \nu(S(\underline{q})-\underline{\theta}\underline{q}-\underline{U})+(1-\nu)(S(\bar{q})-\bar{\theta}\bar{q}-\bar{U}) \\ &\quad +\lambda(\underline{U}-\bar{U}-\Delta\theta\bar{q})+\mu(\nu u(\underline{U})+(1-\nu)u(U)). \end{aligned} \tag{7.59}$$

Optimizing w.r.t. $\underline{U}$ and $\bar{U}$ yields respectively

$$-\nu+\lambda+\mu\nu u'(\underline{U}^{SB})=0 \tag{7.60}$$

$$-(1-\nu)-\lambda+\mu(1-\nu)u'(\bar{U}^{SB})=0. \tag{7.61}$$

Summing the above two equations, we obtain

$$\mu(\nu u'(\underline{U}^{SB}) + (1-\nu)u'(\bar{U}^{SB})) = 1. \tag{7.62}$$

and thus $\mu > 0$. Using (7.62) and inserting it into (7.60) yields

$$\lambda = \frac{\nu(1-\nu)(u'(\bar{U}^{SB}) - u'(\underline{U}^{SB}))}{\nu u'(\underline{U}^{SB}) + (1-\nu)u'(\bar{U}^{SB})}. \tag{7.63}$$

Moreover, (7.20) implies that $\underline{U}^{SB} \geqq \bar{U}^{SB}$ and thus $\lambda \geqq 0$, with $\lambda > 0$ for a positive output $\underline{y}$.

Optimizing with respect to outputs yields respectively

$$S'(\underline{q}^{SB}) = \underline{\theta} \tag{7.64}$$

and

$$S'(\bar{q}^{SB}) = \bar{\theta} + \frac{\lambda}{1-\nu}\Delta\theta. \tag{7.65}$$

Simplifying by using (7.63) yields (7.58).

Thus, with risk aversion, the principal can no longer costlessly structure the agent's information rents to ensure the efficient type's incentive compatibility constraint. Creating a wedge between $\underline{U}$ and $\bar{U}$ to satisfy (7.20) makes the risk-averse agent bear some risk. To guarantee the participation of the risk-averse agent, the principal must now pay a risk premium. Reducing this premium calls for a downward reduction in the inefficient type's output so that the risk borne by the agent is lower. As expected, the agent's risk aversion leads the principal to weaken the incentives.

When the agent becomes infinitely risk averse, everything happens as if he had an ex post individual rationality constraint for the worst state of the world given by (7.23). In the limit, the inefficient agent's output $\bar{q}^{SB}$ and the utility levels $\underline{U}^{SB}$ and $\bar{U}^{SB}$ all converge toward the same solution. So, the previous model at the interim stage can also be interpreted as a model with an ex ante infinitely risk-agent at the zero utility level.

A Risk-Averse Principal

Consider now a risk-averse principal with a Von Neumann-Morgenstern utility function $\nu(\cdot)$ defined on his monetary gains from trade $S(q) - t$ such that $\nu' > 0$, $\nu'' < 0$ and $\nu(0) = 0$. Again, the contract between the principal and the risk-neutral agent is signed before the agent knows his type.

In this context, the first-best contract obviously calls for the first-best output $\underline{q}^*$ and $\bar{q}^*$ being produced. It also calls for the principal to be fully insured between both states of nature and for the agent's ex ante participation constraint to be binding. This leads us to the following two conditions that must be satisfied by the agent's rents $\underline{U}^*$ and $\bar{U}^*$:

$$S(\underline{q}^*) - \underline{\theta}\underline{q}^* - \underline{U}^* = S(\bar{q}^*) - \bar{\theta}\bar{q}^* - \bar{U}^* \tag{7.66}$$

and

$$\nu\underline{U}^* + (1-\nu)\bar{U}^* = 0. \tag{7.67}$$

Solving this system of two equations with two unknowns $(\underline{U}^*, \bar{U}^*)$ yields

$$\underline{U}^* = (1-\nu)(S(\underline{q}^*) - \underline{\theta}\underline{q}^* - (S(\bar{q}^*) - \bar{\theta}\bar{q}^*)) \tag{7.68}$$

and

$$\bar{U}^* = -\nu(S(\underline{q}^*) - \underline{\theta}\underline{q}^* - (S(\bar{q}^*) - \bar{\theta}\bar{q}^*)). \tag{7.69}$$

Note that the first-best profile of information rents satisfies both types' incentive compatibility constraints since

$$\underline{U}^* - \bar{U}^* = S(\underline{q}^*) - \underline{\theta}\underline{q}^* - (S(\bar{q}^*) - \bar{\theta}\bar{q}^*) > \Delta\theta\bar{q}^* \tag{7.70}$$

(from the definition of $\underline{q}^*$) and

$$\bar{U}^* - \underline{U}^* = S(\bar{q}^*) - \bar{\theta}\bar{q}^* - (S(\underline{q}^*) - \underline{\theta}\underline{q}^*) > -\Delta\theta\underline{q}^*, \tag{7.71}$$

(from the definition of $\bar{q}^*$). Hence, the profile of rents $(\underline{U}^*, \bar{U}^*)$ is incentive compatible and the first-best allocation is easily implemented in this framework. We can thus generalize the proposition for the case of risk neutral as follows:

Proposition 7.12.3 *When the principal is risk-averse over the monetary gains $S(q) - t$, the agent is risk-neutral, and contracting takes place ex ante, the optimal incentive contract implements the first-best outcome.*

Remark 7.12.2 It is interesting to note that $\underline{U}^*$ and $\bar{U}^*$ obtained in (7.70) and (7.71) are also the levels of rent obtained in (7.55) and (reftransfer222). Indeed, the lump-sum payment $T^* = \nu(S(\underline{q}^*) - \underline{\theta}\underline{q}^*) + (1-\nu)(S(\bar{q}^*) - \bar{\theta}\bar{q}^*)$, which allows the principal to make the risk-neutral agent residual claimant for the hierarchy's profit, also provides full insurance to the principal. By making the risk-neutral agent the residual claimant for the value of trade, *ex ante* contracting allows the risk-averse principal to get full insurance and implement the first-best outcome despite the informational problem.

Of course this result does not hold anymore if the agent's interim participation constraints must be satisfied. In this case, we still guess a solution such that (7.22) is slack at the optimum. The principal's program now reduces to:

$$\max_{\{(\bar{U},\bar{q});\underline{U},\underline{q})\}} \nu v(S(\underline{q}) - \underline{\theta}\underline{q} - \underline{U}) + (1-\nu)v(S(\bar{q}) - \bar{\theta}\bar{q} - \bar{U})$$

subject to (7.20) to (7.23).

Inserting the values of $\underline{U}$ and $\bar{U}$ that were obtained from the binding constraints in (7.20) and (7.23) into the principal's objective function and optimizing with respect to outputs leads to $\underline{q}^{SB} = \underline{q}^*$, i.e., no distortion for the efficient type, just as in the ease of risk neutrality and a downward distortion of the inefficient type's output $\bar{q}^{SB} < \bar{q}^*$ given by

$$S'(\bar{q}^{SB}) = \bar{\theta} + \frac{\nu v'(\underline{V}^{SB})}{(1-\nu)v'(\bar{V}^{SB})}\Delta\theta. \tag{7.72}$$

where $\underline{V}^{SB} = S(\underline{q}^*) - \underline{\theta}\underline{q}^* - \Delta\theta\bar{q}^{SB}$ and $\bar{V}^{SB} = S(\bar{q}^{SB}) - \bar{\theta}\bar{q}^{SB}$ are the principal's payoffs in both states of nature. We can check that $\bar{V}^{SB} < \underline{V}^{SB}$ since $S(\bar{q}^{SB}) - \underline{\theta}\bar{q}^{SB} < S(\underline{q}^*) - \underline{\theta}\underline{q}^*$ from the definition of $\underline{q}^*$. In particular, we observe that the distortion in the right-hand side of (7.72) is always lower than $\frac{\nu}{1-\nu}\Delta\theta$, its value with a risk-neutral principal. The intuition is straightforward. By increasing $\bar{q}$ above its value with risk neutrality, the risk-averse principal reduces the difference between $\underline{V}^{SB}$ and $\bar{V}^{SB}$. This gives the principal some insurance and increases his ex ante payoff.

For example, if $\nu(x) = \frac{1-e^{-rx}}{r}$, (7.72) becomes $S'(\bar{q}^{SB}) = \bar{\theta} + \frac{\nu}{1-\nu}e^{r(\bar{V}^{SB}-\underline{V}^{SB})}\Delta\theta$. If $r = 0$, we get back the distortion obtained in section 7.7 with a risk-neutral principal and interim participation constraints for the agent. Since $\bar{V}^{SB} < \underline{V}^{SB}$, we observe that the first-best is implemented when r goes to infinity. In the limit, the infinitely risk-averse principal is only interested in the inefficient state of nature for which he wants to maximize the surplus, since there is no rent for the inefficient agent. Moreover, giving a rent to the efficient agent is now without cost for the principal.

Risk aversion on the side of the principal is quite natural in some contexts. A local regulator with a limited budget or a specialized bank dealing with relatively correlated projects may be insufficiently diversified to become completely risk neutral. See Lewis and Sappington (Rand J. Econ, 1995) for an application to the regulation of public utilities.

7.13 Commitment

To solve the incentive problem, we have implicitly assumed that the principal has a strong ability to commit himself not only to a distribution of rents that will induce information revelation but also to some allocative inefficiency designed to reduce the cost of this revelation. Alternatively, this assumption also means that the court of law can perfectly enforce the contract and that neither renegotiating nor reneging on the contract is a feasible alternative for the agent and (or) the principal. What can happen when either of those two assumptions is relaxed?

7.13.1 Renegotiating a Contract

Renegotiation is a voluntary act that should benefit both the principal and the agent. It should be contrasted with a breach of contract, which can hurt one of the contracting parties. One should view a renegotiation procedure as the ability of the contracting partners to achieve a Pareto improving trade if any becomes incentive feasible along the course of actions.

Once the different types have revealed themselves to the principal by selecting the contracts $(\underline{t}^{SB}, \underline{q}^{SB})$ for the efficient type and $(\bar{t}^{SB}, \bar{q}^{SB})$ for the inefficient type, the principal may propose a renegotiation to get around the allocative inefficiency he has imposed on the inefficient agent's output. The gain from this renegotiation comes from raising allocative efficiency for the inefficient type and moving output from $\bar{q}^{SB}$ to $\bar{q}^{*}$. To share these new gains from trade with the inefficient agent, the principal must at least offer him the same utility level as before renegotiation. The participation constraint of the inefficient agent can still be kept at zero when the transfer of this type is raised from $\bar{t}^{SB} = \bar{\theta}\bar{q}^{SB}$ to $\bar{t}^{*} = \bar{\theta}\bar{q}^{*}$. However, raising this transfer also hardens the ex ante incentive compatibility constraint of the efficient type. Indeed, it becomes more valuable for an efficient type to hide his type so that he can obtain this larger transfer, and truthful revelation by the efficient type is no longer obtained in equilibrium. There is a fundamental trade-off between raising efficiency ex post and hardening ex ante incentives when renegotiation is an issue.

7.13.2 Reneging on a Contract

A second source of imperfection arises when either the principal or the agent reneges on their previous contractual obligation. Let us take the case of the principal reneging on the contract. Indeed, once the agent has revealed himself to the principal by selecting the contract within the menu offered by the principal, the latter, having learned the agent's type, might propose the complete information contract which extracts all rents without inducing inefficiency. On the other hand, the agent may want to renege on a contract which gives him a negative ex post utility level as we discussed before. In this case, the threat of the agent reneging a contract signed at the ex ante stage forces the agent's participation constraints to be written in interim terms. Such a setting justifies the focus on the case of interim contracting.

7.14 Informative Signals to Improve Contracting

In this section, we investigate the impacts of various improvements of the principal's information system on the optimal contract. The idea here is to see how signals that are exogenous to the relationship can be used by the principal to better design the contract with the agent.

7.14.1 Ex Post Verifiable Signal

Suppose that the principal, the agent and the court of law observe ex post a viable signal σ which is correlated with θ. This signal is observed after the agent's choice of production. The contract can then be conditioned on both the agent's report and the observed signal that provides useful information on the underlying state of nature.

For simplicity, assume that this signal may take only two values, σ_1 and σ_2. Let the conditional probabilities of these respective realizations of the signal be $\mu_1 = \Pr(\sigma = \sigma_1/\theta = \underline{\theta}) \geqq 1/2$ and $\mu_2 = \Pr(\sigma = \sigma_2/\theta = \bar{\theta}) \geqq 1/2$. Note that, if $\mu_1 = \mu_2 = 1/2$, the signal σ is uninformative. Otherwise, σ_1 brings good news the fact that the agent is efficient and σ_2 brings bad news, since it is more likely that the agent is inefficient in this case.

Let us adopt the following notations for the ex post information rents: $u_{11} = t(\underline{\theta}, \sigma_1) - \underline{\theta} q(\underline{\theta}, \sigma_1)$, $u_{12} = t(\underline{\theta}, \sigma_2) - \underline{\theta} q(\underline{\theta}, \sigma_2)$, $u_{21} = t(\bar{\theta}, \sigma_1) - \bar{\theta} q(\bar{\theta}, \sigma_1)$, and $u_{22} = t(\bar{\theta}, \sigma_2) - \bar{\theta} q(\bar{\theta}, \sigma_2)$. Similar notations are used for the outputs q_{jj}. The agent discovers his type and plays the mechanism before the signal σ realizes. Then the incentive and participation constraints must be written in expectation over the realization of σ. Incentive constraints for both types write respectively as

$$\mu_1 u_{11} + (1 - \mu_1) u_{12} \geqq \mu_1 (u_{21} + \Delta\theta q_{21}) + (1 - \mu_1)(u_{22} + \Delta\theta q_{22}) \tag{7.73}$$

$$(1 - \mu_2) u_{21} + \mu_2 u_{22} \geqq (1 - \mu_2)(u_{11} - \Delta\theta q_{11}) + \mu_2 (u_{12} - \Delta\theta q_{12}). \tag{7.74}$$

Participation constraints for both types are written as

$$\mu_1 u_{11} + (1 - \mu_1) u_{12} \geqq 0, \tag{7.75}$$

$$(1 - \mu_2) u_{21} + \mu_2 u_{22} \geqq 0. \tag{7.76}$$

Note that, for a given schedule of output q_{ij}, the system (7.73) through (7.76) has as many equations as unknowns u_{ij}. When the determinant of the coefficient matrix of the system (7.73) to (7.76) is nonzero, one can find ex post rents u_{ij} (or equivalent transfers) such that all these constraints are binding. In this case, the agent receives no rent whatever his type. Moreover, any choice of production levels, in particular the complete information optimal ones, can be implemented this way. Note that the determinant of the system is nonzero when

$$1 - \mu_1 - \mu_2 \neq 0 \tag{7.77}$$

that fails only if $\mu_1 = \mu_2 = \frac{1}{2}$, which corresponds to the case of an uninformative and useless signal.

7.14.2 Ex Ante Nonverifiable Signal

Now suppose that a nonverifiable binary signal σ about θ is available to the principal at the ex ante stage. Before offering an incentive contract, the principal computes, using the Bayes law, his posterior belief that the agent is efficient for each value of this signal, namely

$$\hat{\nu}_1 = Pr(\theta = \underline{\theta}/\sigma = \sigma_1) = \frac{\nu\mu_1}{\nu\mu_1 + (1-\nu)(1-\mu_2)}, \quad (7.78)$$

$$\hat{\nu}_2 = Pr(\theta = \underline{\theta}/\sigma = \sigma_2) = \frac{\nu(1-\mu_1)}{\nu(1-\mu_1) + (1-\nu)\mu_2}. \quad (7.79)$$

Then the optimal contract entails a downward distortion of the inefficient agents production $\bar{q}^{SB}(\sigma_i))$ which is for signals σ_1, and σ_2 respectively:

$$S'(\bar{q}^{SB}(\sigma_1)) = \bar{\theta} + \frac{\widehat{\nu}_1}{1-\widehat{\nu}_1}\Delta\theta = \bar{\theta} + \frac{\nu\mu_1}{(1-\nu)(1-\mu_2)}\Delta\theta \quad (7.80)$$

$$S'(\bar{q}^{SB}(\sigma_2)) = \bar{\theta} + \frac{\widehat{\nu}_2}{1-\widehat{\nu}_2}\Delta\theta = \bar{\theta} + \frac{\nu(1-\mu_1)}{(1-\nu)\mu_2)}\Delta\theta. \quad (7.81)$$

In the case where $\mu_1 = \mu_2 = \mu > \frac{1}{2}$, we can interpret μ as an index of the informativeness of the signal. Observing σ_1 , the principal thinks that it is more likely that the agent is efficient. A stronger reduction in $\bar{q}^{SB}$and thus in the efficient type's information rent is called for after σ_1. (7.80) shows that incentives decrease with respect to the case without informative signal since $\left(\frac{\mu}{1-\mu} > 1\right)$. In particular, if μ is large enough, the principal shuts down the inefficient firm after having observed σ_1. The principal offers a high-powered incentive contract only to the efficient agent, which leaves him with no rent. On the contrary, because he is less likely to face an efficient type after having observed σ_2, the principal reduces less of the information rent than in the case without an informative signal since $\left(\frac{1-\mu}{\mu} < 1\right)$. Incentives are stronger.

7.15 Contract Theory at Work

This section proposes several classical settings where the basic model of this chapter is useful. Introducing adverse selection in each of these contexts has proved to be a significative improvement of standard microeconomic analysis.

7.15.1 Regulation

In the Baron and Myerson (Econometrica, 1982) regulation model, the principal is a regulator who maximizes a weighted average of the agents' surplus $S(q) - t$ and of a regulated monopoly's profit $U = t - \theta q$, with a weight $\alpha < 1$ for the firms profit. The principal's objective function is written now as $V = S(q) - \theta q - (1-\alpha)U$. Because $\alpha < 1$, it is socially costly to give up a rent to the firm. Maximizing expected social welfare under incentive and participation constraints leads to $\underline{q}^{SB} = \underline{q}^*$ for the efficient type and a downward distortion for the inefficient type, $\bar{q}^{SB} < \bar{q}^*$ which is given by

$$S'(\bar{q}^{SB}) = \bar{\theta} + \frac{\nu}{1-\nu}(1-\alpha)\Delta\theta. \tag{7.82}$$

Note that a higher value of α reduces the output distortion, because the regulator is less concerned by the distribution of rents within society as α increases. If $\alpha = 1$, the firm's rent is no longer costly and the regulator behaves as a pure efficiency maximizer implementing the first-best output in all states of nature.

The regulation literature of the last fifteen years has greatly improved our understanding of government intervention under asymmetric information. We refer to the book of Laffont and Tirole (1993) for a comprehensive view of this theory and its various implications for the design of real world regulatory institutions.

7.15.2 Nonlinear Pricing by a Monopoly

In Maskin and Riley (Rand J. of Economics, 1984), the principal is the seller of a private good with production cost cq who faces a continuum of buyers. The principal has thus a utility function $V = t - cq$. The tastes of a buyer for the private good are such that his utility function is $U = \theta u(q) - t$, where q is the quantity consumed and t his payment to the principal. Suppose that the parameter θ of each buyer is drawn independently from the same distribution on $\Theta = \{\underline{\theta}, \bar{\theta}\}$ with respective probabilities $1-\nu$ and ν.

We are now in a setting with a continuum of agents. However, it is mathematically equivalent to the framework with a single agent. Now ν is the frequency of type $\underline{\theta}$ by the Law of Large Numbers.

Incentive and participation constraints can as usual be written directly in terms of the information rents $\underline{U} = \underline{\theta}u(\underline{q}) - \underline{t}$ and $\bar{U} = \bar{\theta}u(\bar{q}) - \bar{t}$ as

$$\underline{U} \geqq \bar{U} - \Delta\theta u(\bar{q}), \tag{7.83}$$

$$\bar{U} \geqq \underline{U} + \Delta\theta u(\underline{q}), \tag{7.84}$$

$$\underline{U} \geqq 0, \tag{7.85}$$

$$\bar{U} \geqq 0. \tag{7.86}$$

The principal's program now takes the following form:

$$\max_{\{(\bar{U},\bar{q});(\underline{U},\underline{q})\}} v(\bar{\theta}u(\bar{q}) + (1-v)(\underline{\theta}u(\underline{q}) - c\underline{q}) - (\nu\bar{U} + (1-\nu)\underline{U})$$

subject to (7.83) to (7.86).

The analysis is the mirror image of that of the standard model discussed before, where now the efficient type is the one with the highest valuation for the good $\bar{\theta}$. Hence, (7.84) and (7.85) are the two binding constraints. As a result, there is no output distortion with respect to the first-best outcome for the high valuation type and $\bar{q}^{SB} = \bar{q}^*$, where $\bar{\theta}u'(\bar{q}^*) = c$. However, there exists a downward distortion of the low valuation agent's output with respect to the first-best outcome. We have $\underline{q}^{SB} < \underline{q}^*$, where

$$\left(\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta\right) u'(\underline{q}^{SB}) = C \quad \text{and} \quad \underline{\theta}u'(\underline{q}^*) = c. \tag{7.87}$$

7.15.3 Quality and Price Discrimination

Mussa and Rosen (JET, 1978) studied a very similar problem to the nonlinear pricing, where agents buy one unit of a commodity with quality q but are vertically differentiated with respect to their preferences for the good. The marginal cost (and average cost) of producing one unit of quality q is $C(q)$ and the principal has the utility function $V = t - C(q)$. The utility function of an agent is now $U = \theta q - t$ with θ in $\Theta = \{\underline{\theta}, \bar{\theta}\}$, with respective probabilities $1-\nu$ and ν.

Incentive and participation constraints can still be written directly in terms of the information rents $\underline{U} = \underline{\theta}\underline{q} - \underline{t}$ and $\bar{U} = \bar{\theta}\bar{q} - \bar{t}$ as

$$\underline{U} \geqq \bar{U} - \Delta\theta\bar{q}, \tag{7.88}$$

$$\bar{U} \geqq \underline{U} + \Delta\theta\underline{q}, \tag{7.89}$$

$$\underline{U} \geqq 0, \tag{7.90}$$

$$\bar{U} \geqq 0. \tag{7.91}$$

The principal solves now:

$$\max_{\{(\underline{U},\underline{q});(\bar{U},\bar{q})\}} v(\bar{\theta}\bar{q} - C(\bar{q})) + (1-\nu)(\underline{\theta}\underline{q} - C(\underline{q})) - (\nu\bar{U} + (1-\nu)\underline{U})$$

subject to (7.88) to (7.91).

Following procedures similar to what we have done so far, only (7.89) and (7.90) are binding constraints. Finally, we find that the high valuation agent receives the first-best quality $\bar{q}^{SB} = \bar{q}^*$ where $\bar{\theta} = C'(\bar{q}^*)$. However, quality is now reduced below the first-best for the low valuation agent. We have $\underline{q}^{SB} < \underline{q}^*$,where

$$\underline{\theta} = C'(\underline{q}^{SB}) + \frac{\nu}{1-\nu}\Delta\theta \quad \text{and} \quad \underline{\theta} = C'(\underline{q}^*) \tag{7.92}$$

Interestingly, the spectrum of qualities is larger under asymmetric information than under complete information. This incentive of the seller to put a low quality good on the market is a well-documented phenomenon in the industrial organization literature. Some authors have even argued that damaging its own goods may be part of the firm's optimal selling strategy when screening the consumers' willingness to pay for quality is an important issue.

7.15.4 Financial Contracts

Asymmetric information significantly affects the financial markets. For instance, in a paper by Freixas and Laffont (1990), the principal is a lender who provides a loan of size k to a borrower. Capital costs Rk to the lender since it could be invested elsewhere in the economy to earn the risk-free interest rate R. The lender has thus a utility function $V = t - Rk$. The borrower makes a profit $U = \theta f(k) - t$ where $\theta f(k)$ is the production with k units of capital and t is the borrowers repayment to the lender. We assume that $f' > 0$ and $f'' < 0$. The parameter θ is a productivity shock drawn from $\Theta = \{\underline{\theta}, \bar{\theta}\}$ with respective probabilities $1 - \nu$ and ν.

Incentive and participation constraints can again be written directly in terms of the borrower's information rents $\underline{U} = \underline{\theta} f(\underline{k}) - \underline{t}$ and $\bar{U} = \bar{\theta} f(\bar{k}) - \bar{t}$ as

$$\underline{U} \geqq \bar{U} - \Delta\theta f(\bar{k}), \tag{7.93}$$

$$\bar{U} \geqq \underline{U} + \Delta\theta f(\underline{k}), \tag{7.94}$$

$$\underline{U} \geqq 0, \tag{7.95}$$

$$\bar{U} \geqq 0. \tag{7.96}$$

The principal's program takes now the following form:

$$\max_{\{(\underline{U},\underline{k});(\bar{U},\bar{k})\}} v(\bar{\theta}f(\bar{k}) - R\bar{k}) + (1-\nu)(\underline{\theta}f(\underline{k})) - R\underline{k}) - (\nu\bar{U} + (1-\nu)\underline{U})$$

subject to (7.93) to (7.96).

One can check that (7.94) and (7.95) are now the two binding constraints. As a result, there is no capital distortion with respect to the first-best outcome for the high productivity type and $\bar{k}^{SB} = k^*$where $\bar{\theta}f'(\bar{k}^*) = R$. In this case, the return on capital is equal to the risk-free interest rate. However, there also exists a downward distortion in the size of the loan given to a low productivity borrower with respect to the first-best outcome. We have $\underline{k}^{SB} < \underline{k}^*$ where

$$\left(\underline{\theta} - \frac{\nu}{1-\nu}\Delta\theta\right) f'(\underline{k}^{SB}) = R \quad \text{and} \quad \underline{\theta}f'(\underline{k}^*) = R. \tag{7.97}$$

7.15.5 Labor Contracts

Asymmetric information also undermines the relationship between a worker and the firm for which he works. In Green and Kahn (QJE, 1983) and Hart (RES, 1983), the principal is a union (or a set of workers) providing its labor force l to a firm.

The firm makes a profit $\theta f(l) - t$, where $f(l)$ is the return on labor and t is the worker's payment. We assume that $f' > 0$ and $f'' < 0$. The parameter θ is a productivity shock drawn from $\Theta = \{\underline{\theta}, \bar{\theta}\}$ with respective probabilities $1-\nu$ and ν. The firm's objective is to maximize its profit $U = \theta f(l) - t$. Workers have a utility function defined on consumption and labor. If their disutility of labor is counted in monetary terms and all revenues from the firm are consumed, they get $V = v(t - l)$ where l is their disutility of providing l units of labor and $v(\cdot)$ is increasing and concave ($v' > 0$, $v'' < 0$).

In this context, the firm's boundaries are determined before the realization of the shock and contracting takes place ex ante. It should be clear that the model is similar to the one with a risk-averse principal and a risk- neutral agent. So, we know that the risk-averse union will propose a contract to the risk-neutral firm which provides full insurance and implements the first-best levels of employments $\bar{l}$ and $\underline{l}^*$ defined respectively by $\bar{\theta}f'(\bar{l}^*) = 1$ and $\underline{\theta}f'(\underline{l}^*) = 1$.

When workers have a utility function exhibiting an income effect, the analysis will become much harder even in two-type models. For details, see Laffont and Martimort

(2002).

7.16 The Optimal Contract with a Continuum of Types

In this section, we give a brief account of the continuum type case. Most of the principal-agent literature is written within this framework.

Reconsider the standard model with θ in $\Theta = [\underline{\theta}, \bar{\theta}]$, with a cumulative distribution function $F(\theta)$ and a density function $f(\theta) > 0$ on $[\underline{\theta}, \bar{\theta}]$. Since the revelation principle is still valid with a continuum of types, and we can restrict our analysis to direct revelation mechanisms $\{(q(\tilde{\theta}), t(\tilde{\theta}))\}$, which are truthful, i.e., such that

$$t(\theta) - \theta q(\theta) \geqq t(\tilde{\theta}) - \theta q(\tilde{\theta}) \quad \text{for any} \;\; (\theta, \tilde{\theta}) \in \Theta^2. \tag{7.98}$$

In particular, (7.98) implies

$$\begin{aligned} t(\theta) - \theta q(\theta) &\geqq t(\theta') - \theta q(\theta'), & (7.99)\\ t(\theta') - \theta' q(\theta') &\geqq t(\theta) - \theta' q(\theta) \quad \text{for all pairs} \;\; (\theta, \theta') \in \Theta^2. & (7.100)\end{aligned}$$

Adding (7.99) and (7.100) we obtain

$$(\theta - \theta')(q(\theta') - q(\theta)) \geqq 0. \tag{7.101}$$

Thus, incentive compatibility alone requires that the schedule of output $q(\cdot)$ has to be nonincreasing. This implies that $q(\cdot)$ is differentiable almost everywhere. So we will restrict the analysis to differentiable functions.

(7.98) implies that the following first-order condition for the optimal response $\tilde{\theta}$ chosen by type θ is satisfied

$$\dot{t}(\tilde{\theta}) - \theta \dot{q}(\tilde{\theta}) = 0. \tag{7.102}$$

For the truth to be an optimal response for all θ, it must be the case that

$$\dot{t}(\theta) - \theta \dot{q}(\theta) = 0, \tag{7.103}$$

and (7.103) must hold for all θ in Θ since θ is unknown to the principal.

It is also necessary to satisfy the local second-order condition,

$$\ddot{t}(\tilde{\theta})|_{\tilde{\theta}=\theta} - \theta \ddot{q}(\tilde{\theta})|_{\tilde{\theta}=\theta} \leqq 0 \tag{7.104}$$

or

$$\ddot{t}(\theta) - \theta\ddot{q}(\theta) \leqq 0. \tag{7.105}$$

But differentiating (7.103), (7.105) can be written more simply as

$$-\dot{q}(\theta) \geqq 0. \tag{7.106}$$

(7.103) and (7.106) constitute the local incentive constraints, which ensure that the agent does not want to lie locally. Now we need to check that he does not want to lie globally either, therefore the following constraints must be satisfied

$$t(\theta) - \theta q(\theta) \geqq t(\tilde{\theta}) - \theta q(\tilde{\theta}) \quad \text{for any } (\theta, \tilde{\theta}) \in \Theta^2. \tag{7.107}$$

From (7.103) we have

$$t(\theta) - t(\tilde{\theta}) = \int_{\tilde{\theta}}^{\theta} \tau\dot{q}(\tau)d\tau = \theta q(\theta) - \tilde{\theta}q(\tilde{\theta}) - \int_{\tilde{\theta}}^{\theta} q(\tau)d\tau \tag{7.108}$$

or

$$t(\theta) - \theta q(\theta) = t(\tilde{\theta}) - \theta q(\tilde{\theta}) + (\theta - \tilde{\theta})q(\tilde{\theta}) - \int_{\tilde{\theta}}^{\theta} q(\tau)d\tau, \tag{7.109}$$

where $(\theta - \tilde{\theta})q(\tilde{\theta}) - \int_{\tilde{\theta}}^{\theta} q(\tau)d\tau \geqq 0$, because $q(\cdot)$ is nonincreasing.

So, it turns out that the local incentive constraints (7.103) also imply the global incentive constraints.

In such circumstances, the infinity of incentive constraints (7.107) reduces to a differential equation and to a monotonicity constraint. Local analysis of incentives is enough. Truthful revelation mechanisms are then characterized by the two conditions (7.103) and (7.106).

Let us use the rent variable $U(\theta) = t(\theta) - \theta q(\theta)$. The local incentive constraint is now written as (by using (7.103))

$$\dot{U}(\theta) = -q(\theta). \tag{7.110}$$

The optimization program of the principal becomes

$$\max_{\{(U(\cdot), q(\cdot))\}} \int_{\underline{\theta}}^{\bar{\theta}} (S(q(\theta)) - \theta q(\theta) - U(\theta))f(\theta)d\theta \tag{7.111}$$

subject to

$$\dot{U}(\theta) = -q(\theta), \tag{7.112}$$

$$\dot{q}(\theta) \leqq 0, \tag{7.113}$$

$$U(\theta) \geqq 0. \tag{7.114}$$

Using (7.110), the participation constraint (7.114) simplifies to $U(\bar{\theta}) \geqq 0$. As in the discrete case, incentive compatibility implies that only the participation constraint of the most inefficient type can be binding. Furthermore, it is clear from the above program that it will be binding. i.e., $U(\bar{\theta}) = 0$.

Momentarily ignoring (7.113), we can solve (7.112)

$$U(\bar{\theta}) - U(\theta) = -\int_{\theta}^{\bar{\theta}} q(\tau)d\tau \tag{7.115}$$

or, since $U(\bar{\theta}) = 0$,

$$U(\theta) = \int_{\theta}^{\bar{\theta}} q(\tau)d\tau \tag{7.116}$$

The principal's objective function becomes

$$\int_{\underline{\theta}}^{\bar{\theta}} \left(S(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\bar{\theta}} q(\tau)d\tau \right) f(\theta)d\theta, \tag{7.117}$$

which, by an integration of parts, gives

$$\int_{\underline{\theta}}^{\bar{\theta}} \left(S(q(\theta)) - \left(\theta + \frac{F(\theta)}{f(\theta)} \right) q(\theta) \right) f(\theta)d\theta. \tag{7.118}$$

Maximizing pointwise (7.118), we get the second-best optimal outputs

$$S'(q^{SB}(\theta)) = \theta + \frac{F(\theta)}{f(\theta)}, \tag{7.119}$$

which is the first order condition for the case of a continuum of types.

If the monotone hazard rate property $\frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) \geqq 0$ holds, the solution $q^{SB}(\theta)$ of (7.119) is clearly decreasing, and the neglected constraint (7.113) is satisfied. All types choose therefore different allocations and there is no bunching in the optimal contract.

From (7.119), we note that there is no distortion for the most efficient type (since $F(\underline{\theta}) = 0$ and a downward distortion for all the other types.

All types, except the least efficient one, obtain a positive information rent at the optimal contract

$$U^{SB}(\theta) = \int_{\theta}^{\bar{\theta}} q^{SB}(\tau)d\tau. \tag{7.120}$$

Finally, one could also allow for some shutdown of types. The virtual surplus $S(q) - \left(\theta + \frac{F(\theta)}{f(\theta)}\right) q$ decreases with θ when the monotone hazard rate property holds, and shutdown (if any) occurs on an interval $[\theta^*, \bar{\theta}]$. θ^* is obtained as a solution to

$$\max_{\{\theta^*\}} \int_{\underline{\theta}}^{\theta^*} \left(S(q^{SB}(\theta)) - \left(\theta + \frac{F(\theta)}{f(\theta)}\right) q^{SB}(\theta) \right) f(\theta) d\theta.$$

For an interior optimum, we find that

$$S(q^{SB}(\theta^*)) = \left(\theta^* + \frac{F(\theta^*)}{f(\theta^*)}\right) q^{SB}(\theta^*).$$

As in the discrete case, one can check that the Inada condition $S'(0) = +\infty$ and the condition $\lim_{q \to 0} S'(q)q = 0$ ensure the corner solution $\theta^* = \bar{\theta}$.

Remark 7.16.1 The optimal solution above can also be derived by using the Pontryagin principle. The Hamiltonian is then

$$H(q, U, \mu, \theta) = (S(q) - \theta q - U) f(\theta) - \mu q, \tag{7.121}$$

where μ is the co-state variable, U the state variable and q the control variable,

From the Pontryagin principle,

$$\dot{\mu}(\theta) = -\frac{\partial H}{\partial U} = f(\theta). \tag{7.122}$$

From the transversatility condition (since there is no constraint on $U(\cdot)$ at $\underline{\theta}$),

$$\mu(\underline{\theta}) = 0. \tag{7.123}$$

Integrating (7.122) using (7.123), we get

$$\mu(\theta) = F(\theta). \tag{7.124}$$

Optimizing with respect to $q(\cdot)$ also yields

$$S'\left(q^{SB}(\theta)\right) = \theta + \frac{\mu(\theta)}{f(\theta)}, \tag{7.125}$$

and inserting the value of $\mu(\theta)$ obtained from (7.124) again yields(7.119).

We have derived the optimal truthful direct revelation mechanism $\{\left(q^{SB}(\theta), U^{SB}(\theta)\right)\}$ or $\{(q^{SB}(\theta), t^{SB}(\theta))\}$. It remains to be investigated if there is a simple implementation of this mechanism. Since $q^{SB}(\cdot)$ is decreasing, we can invert this function and obtain $\theta^{SB}(q)$. Then,

$$t^{SB}(\theta) = U^{SB}(\theta) + \theta q^{SB}(\theta) \tag{7.126}$$

becomes

$$T(q) = t^{SB}(\theta^{SB}(q)) = \int_{\theta(q)}^{\bar{\theta}} q^{SB}(\tau)d\tau + \theta(q)q. \tag{7.127}$$

To the optimal truthful direct revelation mechanism we have associated a nonlinear transfer $T(q)$. We can check that the agent confronted with this nonlinear transfer chooses the same allocation as when he is faced with the optimal revelation mechanism. Indeed, we have $\frac{d}{dq}(T(q) - \theta q) = T'(q) - \theta = \frac{dt^{SB}}{d\theta} \cdot \frac{d\theta^{SB}}{dq} - \theta = 0$, since $\frac{dt^{SB}}{d\theta} - \theta \frac{dq^{SB}}{d\theta} = 0$.

To conclude, the economic insights obtained in the continuum case are not different from those obtained in the two-state case.

7.17 Further Extensions

The main theme of this chapter was to determine how the fundamental conflict between rent extraction and efficiency could be solved in a principal-agent relationship with adverse selection. In the models discussed, this conflict was relatively easy to understand because it resulted from the simple interaction of a single incentive constraint with a single participation constraint. Here we would mention some possible extensions.

One can consider a straightforward three-type extension of the standard model. One can also deal with a bidimensional adverse selection model, a two-type model with type-dependent reservation utilities, random participation constraints, the limited liability constraints, and the audit models. For detailed discussion about these topics and their applications, see Laffont and Martimort (2002).

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Chapter 8

Moral Hazard: The Basic Trade-Offs

8.1 Introduction

In the previous chapter, we stressed that the delegation of tasks creates an information gap between the principal and his agent when the latter learns some piece of information relevant to determining the efficient volume of trade. Adverse selection is not the only informational problem one can imagine. Agents may also choose actions that affect the value of trade or, more generally, the agent's performance. The principal often loses any ability to control those actions that are no longer observable, either by the principal who offers the contract or by the court of law that enforces it. In such cases we will say that there is moral hazard.

The leading candidates for such moral hazard actions are effort variables, which positively influence the agent's level of production but also create a disutility for the agent. For instance the yield of a field depends on the amount of time that the tenant has spent selecting the best crops, or the quality of their harvesting. Similarly, the probability that a driver has a car crash depends on how safely he drives, which also affects his demand for insurance. Also, a regulated firm may have to perform a costly and nonobservable investment to reduce its cost of producing a socially valuable good.

As in the case of adverse selection, asymmetric information also plays a crucial role in the design of the optimal incentive contract under moral hazard. However, instead of being an exogenous uncertainty for the principal, uncertainty is now endogenous. The probabilities of the different states of nature, and thus the expected volume of trade, now depend explicitly on the agent's effort. In other words, the realized production level is only a noisy signal of the agent's action. This uncertainty is key to understanding the contractual problem under moral hazard. If the mapping between effort and performance were completely deterministic, the principal and the court of law would have no difficulty in inferring the agent's effort from the observed output. Even if the agent's effort was not observable directly, it could be indirectly contracted upon, since output would itself be observable and verifiable.

We will study the properties of incentive schemes that induce a positive and costly effort. Such schemes must thus satisfy an incentive constraint and the agent's participation constraint. Among such schemes, the principal prefers the one that implements the positive level of effort at minimal cost. This cost minimization yields the characterization of the second-best cost of implementing this effort. In general, this second-best cost is greater than the first-best cost that would be obtained by assuming that effort is observable. An allocative inefficiency emerges as the result of the conflict of interests between the principal and the agent.

8.2 The Model

8.2.1 Effort and Production

We consider an agent who can exert a costly effort e. Two possible values can be taken by e, which we normalize as a zero effort level and a positive effort of one: e in $\{0, 1\}$. Exerting effort e implies a disutility for the agent that is equal to $\psi(e)$ with the normalization $\psi(0) = \psi_0 = 0$ and $\psi_1 = \psi$.

The agent receives a transfer t from the principal. We assume that his utility function is separable between money and effort, $U = u(t) - \psi(e)$, with $u(\cdot)$ increasing and concave ($u' > 0, u'' < 0$). Sometimes we will use the function $h = u^{-1}$, the inverse function of $u(\cdot)$, which is increasing and convex ($h' > 0, h'' > 0$).

Production is stochastic, and effort affects the production level as follows: the stochastic production level $\tilde{q}$ can only take two values $\{\underline{q}, \bar{q}\}$, with $\bar{q} - \underline{q} = \Delta q > 0$, and the stochastic influence of effort on production is characterized by the probabilities $\Pr(\tilde{q} = \bar{q}|e = 0) = \pi_0$, and $\Pr(\tilde{q} = \bar{q}|e = 1) = \pi_1$, with $\pi_1 > \pi_0$. We will denote the difference between these two probabilities by $\Delta\pi = \pi_1 - \pi_0$.

Note that effort improves production in the sense of first-order stochastic dominance, i.e., $\Pr(\tilde{q} \leqq q^*|e)$ is decreasing with e for any given production q^*. Indeed, we have $\Pr(\tilde{q} \leqq \underline{q}|e = 1) = 1 - \pi_1 < 1 - \pi_0 = \Pr(\tilde{q} \leqq \underline{q}|e = 0)$ and $\Pr(\tilde{q} \leqq \bar{q}|e = 1) = 1 = \Pr(\tilde{q} \leqq \bar{q}|e = 0)$

8.2.2 Incentive Feasible Contracts

Since the agent's action is not directly observable by the principal, the principal can only offer a contract based on the observable and verifiable production level. i.e., a function $\{t(\tilde{q})\}$ linking the agent's compensation to the random output $\tilde{q}$. With two possible outcomes $\bar{q}$ and $\underline{q}$, the contract can be defined equivalently by a pair of transfers $\bar{t}$ and $\underline{t}$. Transfer $\bar{t}$ (resp. $\underline{t}$) is the payment received by the agent if the production $\bar{q}$ (resp. $\underline{q}$) is realized.

The risk-neutral principal's expected utility is now written as

$$V_1 = \pi_1(S(\bar{q}) - (\bar{t}) + (1 - \pi_1)(S(\underline{q}) - \underline{t}) \tag{8.1}$$

if the agent makes a positive effort ($e = 1$) and

$$V_0 = \pi_0(S(\bar{q}) - (\bar{t}) + (1 - \pi_0)(S(\underline{q}) - \underline{t}) \tag{8.2}$$

if the agent makes no effort ($e = 0$). For notational simplicity, we will denote the principal's benefits in each state of nature by $S(\bar{q}) = \bar{S}$ and $S(\underline{q}) = \underline{S}$.

Each level of effort that the principal wishes to induce corresponds to a set of contracts ensuring moral hazard incentive constraint and participation constraint are satisfied:

$$\pi_1 u(\bar{t}) + (1 - \pi_1)u(\underline{t}) - \psi \;\geqq\; \pi_0 u(\bar{t}) + (1 - \pi_0)u(\underline{t}) \tag{8.3}$$

$$\pi_1 u(\bar{t}) + (1 - \pi_1)u(\underline{t}) - \psi \;\geqq\; 0. \tag{8.4}$$

Note that the participation constraint is ensured at the ex ante stage, i.e., before the realization of the production shock.

Definition 8.2.1 *An incentive feasible contract satisfies the incentive and participation constraints (8.3) and (8.4).*

The timing of the contracting game under moral hazard is summarized in the figure below.

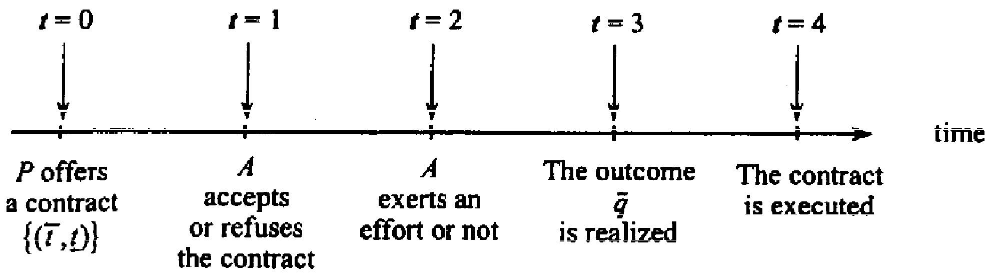

Figure 8.1: Timing of contracting under moral harzard.

8.2.3 The Complete Information Optimal Contract

As a benchmark, let us first assume that the principal and a benevolent court of law can both observe effort. Then, if he wants to induce effort, the principal's problem becomes

$$\max_{\{(\bar{t},\underline{t})\}} \pi_1(\bar{S}-\bar{t})+(1-\pi_1)(\underline{S}-\underline{t}) \tag{8.5}$$

subject to (8.4).

Indeed, only the agents participation constraint matters for the principal, because the agent can be forced to exert a positive level of effort. If the agent were not choosing this level of effort, the agent could be heavily punished, and the court of law could commit to enforce such a punishment.

Denoting the multiplier of this participation constraint by λ and optimizing with respect to $\bar{t}$ and $\underline{t}$ yields, respectively, the following first-order conditions:

$$-\pi_1+\lambda\pi_1 u'(\bar{t}^*)=0, \tag{8.6}$$

$$-(1-\pi_1)+\lambda(1-\pi_1)u'(\underline{t}^*)=0, \tag{8.7}$$

where $\bar{t}^*$ and $\underline{t}^*$ are the first-best transfers.

From (8.6) and (8.7) we immediately derive that $\lambda=\frac{1}{u'(\underline{t}^*)}=\frac{1}{u'(\bar{t}^*)}>0$, and finally that $t^*=\bar{t}^*=\underline{t}^*$.

Thus, with a verifiable effort, the agent obtains full insurance from the risk-neutral principal, and the transfer t^* he receives is the same whatever the state of nature. Because the participation constraint is binding we also obtain the value of this transfer, which is just enough to cover the disutility of effort, namely $t^*=h(\psi)$. This is also the expected payment made by the principal to the agent, or the first-best cost C^{FB} of implementing the positive effort level.

For the principal, inducing effort yields an expected payoff equal to

$$V_1 = \pi_1 \bar{S} + (1 - \pi_1)\underline{S} - h(\psi) \tag{8.8}$$

Had the principal decided to let the agent exert no effort, e_0, he would make a zero payment to the agent whatever the realization of output. In this scenario, the principal would instead obtain a payoff equal to

$$V_0 = \pi_0 \bar{S} + (1 - \pi_0)\underline{S}. \tag{8.9}$$

Inducing effort is thus optimal from the principal's point of view when $V_1 \geqq V_0$, i.e., $\pi_1 \bar{S} + (1 - \pi_1)\underline{S} - h(\psi) \geqq \pi_0 \bar{S} + (1 - \pi_0)\underline{S}$, or to put it differently, when the expected gain of effect is greater than first-best cost of inducing effect, i.e.,

$$\underbrace{\Delta\pi\Delta S} \geqq \underbrace{h(\psi)} \tag{8.10}$$

where $\Delta S = \bar{S} - \underline{S} > 0$.

Denoting the benefit of inducing a strictly positive effort level by $B = \Delta\pi\Delta S$, the first-best outcome calls for $e^* = 1$ if and only if $B > h(\psi)$, as shown in the figure below.

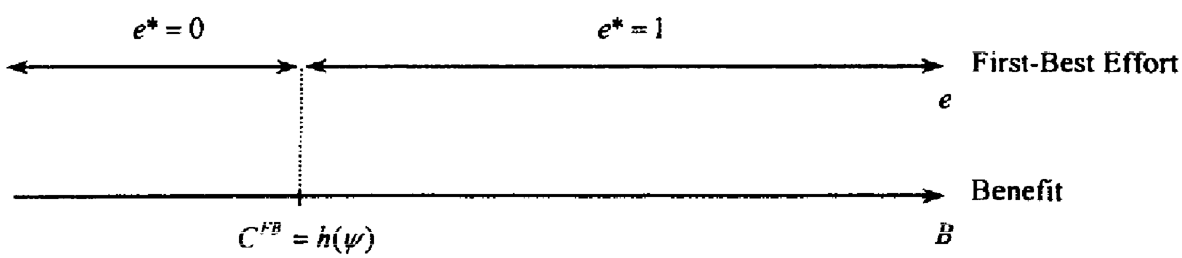

Figure 8.2: First-best level of effort.

8.3 Risk Neutrality and First-Best Implementation

If the agent is risk-neutral, we have (up to an affine transformation) $u(t) = t$ for all t and $h(u) = u$ for all u. The principal who wants to induce effort must thus choose the contract that solves the following problem:

$$\max_{\{(\bar{t}, \underline{t})\}} \pi_1(\bar{S} - \bar{t}) + (1 - \pi_1)(\underline{S} - \underline{t})$$

$$\pi_1 \bar{t} + (1 - \pi_1)\underline{t} - \psi \geqq \pi_0 \bar{t} + (1 - \pi_0)\underline{t} \tag{8.11}$$

$$\pi_1 \bar{t} + (1-\pi_1)\underline{t} - \psi \geqq 0. \tag{8.12}$$

With risk neutrality the principal can, for instance, choose incentive compatible transfers $\bar{t}$ and $\underline{t}$, which make the agent's participation constraint binding and leave no rent to the agent. Indeed, solving (8.11) and (8.12) with equalities, we immediately obtain

$$\underline{t}^* = -\frac{\pi_0}{\Delta\pi}\psi \tag{8.13}$$

and

$$\bar{t}^* = \frac{1-\pi_0}{\Delta\pi}\psi. \tag{8.14}$$

The agent is rewarded if production is high. His net utility in this state of nature $\bar{U}^* = \bar{t}^* - \psi = \frac{1-\pi_1}{\Delta\pi}\psi > 0$. Conversely, the agent is punished if production is low. His corresponding net utility $\underline{U}^* = \underline{t}^* - \psi = -\frac{\pi_1}{\Delta\pi}\psi < 0$.

The principal makes an expected payment $\pi_1\bar{t}^* + (1-\pi_1)\underline{t}^* = \psi$, which is equal to the disutility of effort he would incur if he could control the effort level perfectly. The principal can costlessly structure the agent's payment so that the latter has the right incentives to exert effort. Using (8.13) and (8.14), his expected gain from exerting effort is thus $\Delta\pi(\bar{t}^* - \underline{t}^*) = \psi$ when increasing his effort from $e = 0$ to $e = 1$.

Proposition 8.3.1 *Moral hazard is not an issue with a risk-neutral agent despite the nonobservability of effort. The first-best level of effort is still implemented.*

Remark 8.3.1 One may find the similarity of these results with those described last chapter. In both cases, when contracting takes place ex ante, the incentive constraint, under either adverse selection or moral hazard, does not conflict with the ex ante participation constraint with a risk-neutral agent, and the first-best outcome is still implemented.

Remark 8.3.2 Inefficiencies in effort provision due to moral hazard will arise when the agent is no longer risk-neutral. There are two alternative ways to model these transaction costs. One is to maintain risk neutrality for positive income levels but to impose a limited liability constraint, which requires transfers not to be too negative. The other is to let the agent be strictly risk-averse. In the following, we analyze these two contractual environments and the different trade-offs they imply.

8.4 The Trade-Off Between Limited Liability Rent Extraction and Efficiency

Let us consider a risk-neutral agent. As we have already seen, (8.3) and (8.4) now take the following forms:

$$\pi_1\bar{t} + (1-\pi_1)\underline{t} - \psi \geqq \pi_0\bar{t} + (1-\pi_0)\underline{t} \tag{8.15}$$

and

$$\pi_1\bar{t} + (1-\pi_1)\underline{t} - \psi \geqq 0. \tag{8.16}$$

Let us also assume that the agent's transfer must always be greater than some exogenous level $-l$, with $l \geqq 0$. Thus, limited liability constraints in both states of nature are written as

$$\bar{t} \geqq -l \tag{8.17}$$

and

$$\underline{t} \geqq -l. \tag{8.18}$$

These constraints may prevent the principal from implementing the first-best level of effort even if the agent is risk-neutral. Indeed, when he wants to induce a high effort, the principal's program is written as

$$\max_{\{(\bar{t},\underline{t})\}} \pi_1(\bar{S} - \bar{t}) + (1-\pi_1)(\underline{S} - \underline{t}) \tag{8.19}$$

subject to (8.15) to (8.18).

Then, we have the following proposition.

Proposition 8.4.1 *With limited liability, the optimal contract inducing effort from the agent entails:*

(1) For $l > \frac{\pi_0}{\Delta\pi}\psi$, only (8.15) and (8.16) are binding. Optimal transfers are given by (8.13) and (8.14). The agent has no expected limited liability rent; $EU^{SB} = 0$.

(2) For $0 \leqq l \leqq \frac{\pi_0}{\Delta\pi}\psi$, (8.15) and (8.18) are binding. Optimal transfers are then given by:

$$\underline{t}^{SB} = -l, \tag{8.20}$$

$$\bar{t}^{SB} = -l + \frac{\psi}{\Delta\pi}. \tag{8.21}$$

(3) Moreover, the agent's expected limited liability rent EU^{SB} is non-negative:

$$EU^{SB} = \pi_1 \bar{t}^{SB} + (1-\pi_1)\underline{t}^{SB} - \psi = -l + \frac{\pi_0}{\Delta\pi}\psi \geqq 0. \tag{8.22}$$

Proof. First suppose that $0 \leqq l \leqq \frac{\pi_0}{\Delta\pi}\psi$.We conjecture that (8.15) and (8.18) are the only relevant constraints. Of course, since the principal is willing to minimize the payments made to the agent, both constraints must be binding. Hence, $\underline{t}^{SB} = -l$ and $\bar{t}^{SB} = -l + \frac{\psi}{\Delta\pi}$. We check that (8.17) is satisfied since $-l + \frac{\psi}{\Delta\pi} > -l$. We also check that (8.16) is satisfied since $\pi_1 \bar{t}^{SB} + (1-\pi_1)\underline{t}^{SB} - \psi = -l + \frac{\pi_0}{\Delta\pi}\psi \geqq 0$.

For $l > \frac{\pi_0}{\Delta\pi}\psi$, note that the transfers $\underline{t}^* = -\frac{\pi_0}{\Delta\pi}\psi$, and $\bar{t}^* = -\psi + \frac{(1-\pi_1)}{\Delta\pi}\psi > \underline{t}^*$ are such that both limited liability constraints (8.17) and (8.18) are strictly satisfied, and (8.15) and (8.16) are both binding. In this case, it is costless to induce a positive effort by the agent, and the first-best outcome can be implemented. The proof is completed.

Note that only the limited liability constraint in the bad state of nature may be binding. When the limited liability constraint (8.18) is binding, the principal is limited in his punishments to induce effort. The risk- neutral agent does not have enough assets to cover the punishment if $\underline{q}$ is realized in order to induce effort provision. The principal uses rewards when a good state of nature $\bar{q}$ is realized. As a result, the agent receives a non-negative ex ante limited liability rent described by (8.22). Compared with the case without limited liability, this rent is actually the additional payment that the principal must incur because of the conjunction of moral hazard and limited liability.

As the agent becomes endowed with more assets, i.e., as l gets larger, the conflict between moral hazard and limited liability diminishes and then disappears whenever l is large enough.

8.5 The Trade-Off Between Insurance and Efficiency

Now suppose the agent is risk-averse. The principal's program is written as:

$$\max_{\{(\bar{t},\underline{t})\}} \pi_1(\bar{S} - \bar{t}) + (1-\pi_1)(\underline{S} - \underline{t}) \tag{8.23}$$

subject to (8.3) and (8.4).

Since the principal's optimization problem may not be a concave program for which the first-order Kuhn and Tucker conditions are necessary and sufficient, we make the following change of variables. Define $\bar{u} = u(\bar{t})$ and $\underline{u} = u(\underline{t})$, or equivalently let $\bar{t} = h(\bar{u})$ and $\underline{t}= h(\underline{u})$. These new variables are the levels of ex post utility obtained by the agent in both states of nature. The set of incentive feasible contracts can now be described by two linear constraints:

$$\pi_1\bar{u} + (1-\pi_1)\underline{u} - \psi \geqq \pi_0\bar{u} + (1-\pi_0)\underline{u}, \tag{8.24}$$

$$\pi_1\bar{u} + (1-\pi_1)\underline{u} - \psi \geqq 0, \tag{8.25}$$

which replaces (8.3) and (8.4), respectively.

Then, the principal's program can be rewritten as

$$\max_{\{(\bar{u},\underline{u})\}} \pi_1(\bar{S} - h(\bar{u})) + (1-\pi_1)(\underline{S} - h(\underline{u})) \tag{8.26}$$

subject to (8.24) and (8.25).

Note that the principal's objective function is now strictly concave in $(\bar{u}, \underline{u})$ because $h(\cdot)$ is strictly convex. The constraints are now linear and the interior of the constrained set is obviously non-empty.

8.5.1 Optimal Transfers

Letting λ and μ be the non-negative multipliers associated respectively with the constraints (8.24) and (8.25), the first-order conditions of this program can be expressed as

$$-\pi_1 h'(\bar{u}^{SB}) + \lambda\Delta\pi + \mu\pi_1 = -\frac{\pi_1}{u'(\bar{t}^{SB})} + \lambda\Delta\pi + \mu\pi_1 = 0, \tag{8.27}$$

$$-(1-\pi_1)h'(\underline{u}^{SB}) - \lambda\Delta\pi + \mu(1-\pi_1) = -\frac{(1-\pi_1)}{u'(\underline{t}^{SB})} - \lambda\Delta\pi + \mu(1-\pi_1) = 0. \tag{8.28}$$

where $\bar{t}^{SB}$ and $\underline{t}^{SB}$ are the second-best optimal transfers. Rearranging terms, we get

$$\frac{1}{u'(\bar{t}^{SB})} = \mu + \lambda\frac{\Delta\pi}{\pi_1}, \tag{8.29}$$

$$\frac{1}{u'(\underline{t}^{SB})} = \mu - \lambda\frac{\Delta\pi}{1-\pi_1}. \tag{8.30}$$

The four variables $(\underline{t}^{SB}, \bar{t}^{SB}, \lambda, \mu)$ are simultaneously obtained as the solutions to the system of four equations (8.24), (8.25), (8.29), and (8.30). Multiplying (8.29) by π_1 and (8.30) by $1-\pi_1$, and then adding those two modified equations we obtain

$$\mu = \frac{\pi_1}{u'(\bar{t}^{SB})} + \frac{1-\pi_1}{u'(\underline{t}^{SB})} > 0. \tag{8.31}$$

Hence, the participation constraint (8.16) is necessarily binding. Using (8.31) and (8.29), we also obtain

$$\lambda = \frac{\pi_1(1-\pi_1)}{\Delta\pi}\left(\frac{1}{u'(\bar{t}^{SB})} - \frac{1}{u'(\underline{t}^{SB})}\right), \tag{8.32}$$

where λ must also be strictly positive. Indeed, from (8.24) we have $\bar{u}^{SB} - \underline{u}^{SB} \geqq \frac{\psi}{\Delta\pi} > 0$ and thus $\bar{t}^{SB} > \underline{t}^{SB}$, implying that the right-hand side of (8.32) is strictly positive since $u'' < 0$. Using that (8.24) and (8.25) are both binding, we can immediately obtain the values of $u(\bar{t}^{SB})$ and $u(\underline{t}^{SB})$ by solving a system of two equations with two unknowns.

Note that the risk-averse agent does not receive full insurance anymore. Indeed, with full insurance, the incentive compatibility constraint (8.3) can no longer be satisfied. Inducing effort requires the agent to bear some risk, the following proposition provides a summary.

Proposition 8.5.1 *When the agent is strictly risk-averse, the optimal contract that induces effort makes both the agent's participation and incentive constraints binding. This contract does not provide full insurance. Moreover, second- best transfers are given by*

$$\bar{t}^{SB} = h\left(\psi + (1-\pi_1)\frac{\psi}{\Delta\pi}\right) \tag{8.33}$$

and

$$\underline{t}^{SB} = h\left(\psi - \pi_1\frac{\psi}{\Delta\pi}\right). \tag{8.34}$$

8.5.2 The Optimal Second-Best Effort

Let us now turn to the question of the second-best optimality of inducing a high effort, from the principal's point of view. The second-best cost C^{SB} of inducing effort under moral hazard is the expected payment made to the agent $C^{SB} = \pi_1\bar{t}^{SB} + (1-\pi_1)\underline{t}^{SB}$. Using (8.33) and (8.34), this cost is rewritten as

$$C^{SB} = \pi_1 h\left(\psi + (1-\pi_1)\frac{\psi}{\Delta\pi}\right) + (1-\pi_1)h\left(\psi - \frac{\pi_1\psi}{\Delta\pi}\right). \tag{8.35}$$

The benefit of inducing effort is still $B = \Delta\pi\Delta S$, and a positive effort $e^* = 1$ is the optimal choice of the principal whenever

$$\Delta\pi\Delta S \geqq C^{SB} = \pi_1 h\left(\psi + (1-\pi_1)\frac{\psi}{\Delta\pi}\right) + (1-\pi_1)h\left(\psi - \frac{\pi_1\psi}{\Delta\pi}\right). \tag{8.36}$$

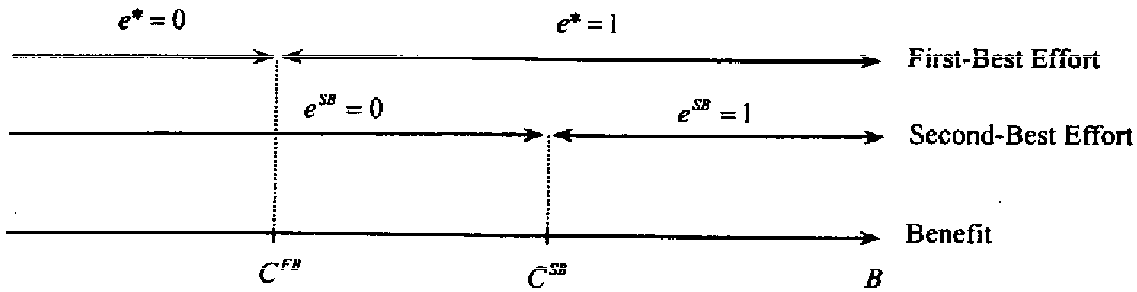

Figure 8.3: Second-best level of effort with moral hazard and risk aversion.

With $h(\cdot)$ being strictly convex, Jensen's inequality implies that the right-hand side of (8.36) is strictly greater than the first-best cost of implementing effort $C^{FB} = h(\psi)$. Therefore, inducing a higher effort occurs less often with moral hazard than when effort is observable. The above figure represents this phenomenon graphically.

For B belonging to the interval $[C^{FB}, C^{SB}]$, the second-best level of effort is zero and is thus strictly below its first- best value. There is now an under-provision of effort because of moral hazard and risk aversion.

Proposition 8.5.2 *With moral hazard and risk aversion, there is a trade-off between inducing effort and providing insurance to the agent. In a model with two possible levels of effort, the principal induces a positive effort from the agent less often than when effort is observable.*

8.6 More than Two Levels of Performance

We now extend our previous 2×2 model to allow for more than two levels of performance. We consider a production process where n possible outcomes can be realized. Those performances can be ordered so that $q_1 < q_2 < \cdots < q_i < q_n$. We denote the principal's return in each of those states of nature by $S_i = S(q_i)$. In this context, a contract is a n-tuple of payments $\{(t_1, \ldots, t_n)\}$. Also, let π_{ik} be the probability that production q_i takes place when the effort level is e_k. We assume that π_{ik} for all pairs (i, k) with $\sum_{i=1}^{n} \pi_{ik} = 1$.

Finally, we keep the assumption that only two levels of effort are feasible. i.e., e_k in $\{0,1\}$. We still denote $\Delta\pi_i = \pi_{i1} - \pi_{i0}$.

8.6.1 Limited Liability

Consider first the limited liability model. If the optimal contract induces a positive effort, it solves the following program:

$$\max_{\{(t_1,\dots,t_n)\}} \sum_{i=1}^{n} \pi_{i1}(S_i - t_i) \tag{8.37}$$

subject to

$$\sum_{i=1}^{n} \pi_{i1} t_i - \psi \geqq 0, \tag{8.38}$$

$$\sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) t_i \geqq \psi, \tag{8.39}$$

$$t_i \geqq 0, \quad \text{for all } i \in \{1,\dots,n\}. \tag{8.40}$$

(8.38) is the agent's participation constraint. (8.39) is his incentive constraint. (8.40) are all the limited liability constraints by assuming that the agent cannot be given a negative payment.

First, note that the participation constraint (8.38) is implied by the incentive (8.39) and the limited liability (8.40) constraints. Indeed, we have

$$\sum_{i=1}^{n} \pi_{i1} t_i - \psi \geqq \underbrace{\sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) t_i - \psi} + \underbrace{\sum_{i=1}^{n} \pi_{i0} t_i} \geqq 0.$$

Hence, we can neglect the participation constraint (8.38) in the optimization of the principal's program.

Denoting the multiplier of (8.39) by λ and the respective multipliers of (8.40) by ξ_i, the first-order conditions lead to

$$-\pi_{i1} + \lambda \Delta\pi_i + \xi_i = 0. \tag{8.41}$$

with the slackness conditions $\xi_i t_i = 0$ for each i in $\{1,\dots,n\}$.

For such that the second-best transfer t_i^{SB} is strictly positive, $\xi_i = 0$, and we must have $\lambda = \frac{\pi_{i1}}{\pi_{i1} - \pi_{i0}}$ for any such i. If the ratios $\frac{\pi_{i1} - \pi_{i0}}{\pi_{i1}}$ all different, there exists a single index j such that $\frac{\pi_{j1} - \pi_{j0}}{\pi_{j1}}$ is the highest possible ratio. The agent receives a strictly positive transfer only in this particular state of nature j, and this payment is such that the incentive constraint (8.39) is binding, i.e., $t_j^{SB} = \frac{\psi}{\pi_{j1}-\pi_{j0}}$. In all other states, the agent receives no transfer and $t_i^{SB} = 0$ for all $i \neq j$. Finally, the agent gets a strictly positive ex ante limited liability rent that is worth $EU^{SB} = \frac{\pi_{j0}\psi}{\pi_{j1}-\pi_{j0}}$.

The important point here is that the agent is rewarded in the state of nature that is the most informative about the fact that he has exerted a positive effort. Indeed, $\frac{\pi_{i1}-\pi_{i0}}{\pi_{i1}}$ can be interpreted as a likelihood ratio. The principal therefore uses a maximum likelihood ratio criterion to reward the agent. The agent is only rewarded when this likelihood ratio is maximized. Like an econometrician, the principal tries to infer from the observed output what has been the parameter (effort) underlying this distribution. But here the parameter is endogenously affected by the incentive contract.

Definition 8.6.1 *The probabilities of success satisfy the monotone likelihood ratio property (MLRP) if* $\frac{\pi_{i1}-\pi_{i0}}{\pi_{i1}}$ *is nondecreasing in* i.

Proposition 8.6.1 *If the probability of success satisfies MLRP, the second-best payment* t_i^{SB} *received by the agent may be chosen to be nondecreasing with the level of production* q_i.

8.6.2 Risk Aversion

Suppose now that the agent is strictly risk-averse. The optimal contract that induces effort must solve the program below:

$$\max_{\{t_1,\dots,t_n)\}} \sum_{i=1}^{n} \pi_{i1}(S_i - t_i) \tag{8.42}$$

subject to

$$\sum_{i=1}^{n} \pi_{i1}u(t_i) - \psi \geqq \sum_{i=1}^{n} \pi_{i0}u(t_i) \tag{8.43}$$

and

$$\sum_{i=1}^{n} \pi_{i1}u(t_i) - \psi \geqq 0, \tag{8.44}$$

where the latter constraint is the agent's participation constraint.

Using the same change of variables as before, it should be clear that the program is again a concave problem with respect to the new variables $u_i = u(t_i)$. Using the same notations as before, the first-order conditions of the principal's program are written as:

$$\frac{1}{u'(t_i^{SB})} = \mu + \lambda \left(\frac{\pi_{i1} - \pi_{i0}}{\pi_{i1}} \right) \quad \text{for all } i \in \{1, \ldots, n\}. \tag{8.45}$$

Multiplying each of these equations by π_{i1} and summing over i yields $\mu = E_q \left(\frac{1}{u'(\tilde{t}_i^{SB})} \right) > 0$, where E_q denotes the expectation operator with respect to the distribution of outputs induced by effort $e = 1$.

Multiplying (8.45) by $\pi_{i1} u(t_i^{SB})$, summing all these equations over i, and taking into account the expression of μ obtained above yields

$$\lambda \left(\sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) u(t_i^{SB}) \right) = E_q \left(u(\tilde{t}_i^{SB}) \left(\frac{1}{u'(\tilde{t}_i^{SB})} - E \left(\frac{1}{u'(\tilde{t}_i^{SB})} \right) \right) \right). \tag{8.46}$$

Using the slackness condition $\lambda \left(\sum_{i=1}^{n} (\pi_{i1} - \pi_{i0}) u(t_i^{SB}) - \psi \right) = 0$ to simplify the left-hand side of (8.46), we finally get

$$\lambda \psi = cov \left(u(\tilde{t}_i^{SB}), \frac{1}{u'(\tilde{t}_i^{SB})} \right). \tag{8.47}$$

By assumption, $u(\cdot)$ and $u'(\cdot)$ covary in opposite directions. Moreover, a constant wage $t_i^{SB} = t^{SB}$ for all i does not satisfy the incentive constraint, and thus t_i^{SB} cannot be constant everywhere. Hence, the right-hand side of (8.47) is necessarily strictly positive. Thus we have $\lambda > 0$, and the incentive constraint is binding.

Coming back to (8.45), we observe that the left-hand side is increasing in t_i^{SB} since $u(\cdot)$ is concave. For t_i^{SB} to be nondecreasing with i, MLRP must again hold. Then higher outputs are also those that are the more informative ones about the realization of a high effort. Hence, the agent should be more rewarded as output increases.

8.7 Contract Theory at Work

This section elaborates on the moral hazard paradigm discussed so far in a number of settings that have been discussed extensively in the contracting literature.

8.7.1 Efficiency Wage

Let us consider a risk-neutral agent working for a firm, the principal. This is a basic model studied by Shapiro and Stiglitz (AER, 1984). By exerting effort e in $\{0, 1\}$, the firm's added value is $\bar{V}$ (resp. $\underline{V}$) with probability $\pi(e)$ (resp. $1-\pi(e)$). The agent can only be rewarded for a good performance and cannot be punished for a bad outcome, since they are protected by limited liability.

To induce effort, the principal must find an optimal compensation scheme $\{(\underline{t},\bar{t})\}$ that is the solution to the program below:

$$\max_{\{(\underline{t},\bar{t})\}} \pi_1(\bar{V}-\bar{t})+(1-\pi_1)(\underline{V}-\underline{t}) \tag{8.48}$$

subject to

$$\pi_1\bar{t}+(1-\pi_1)\underline{t}-\psi \geqq \pi_0\bar{t}+(1-\pi_0)\underline{t}, \tag{8.49}$$

$$\pi_1\bar{t}+(1-\pi_1)\underline{t}-\psi \geqq 0, \tag{8.50}$$

$$\underline{t} \geqq 0. \tag{8.51}$$

The problem is completely isomorphic to the one analyzed earlier. The limited liability constraint is binding at the optimum, and the firm chooses to induce a high effort when $\Delta\pi\Delta V \geqq \frac{\pi_1\psi}{\Delta\pi}$. At the optimum, $\underline{t}^{SB}=0$ and $\bar{t}^{SB}>0$. The positive wage $\bar{t}^{SB}=\frac{\psi}{\Delta\pi}$ is often called an efficiency wage because it induces the agent to exert a high (efficient) level of effort. To induce production, the principal must give up a positive share of the firm's profit to the agent.

8.7.2 Sharecropping

The moral hazard paradigm has been one of the leading tools used by development economists to analyze agrarian economies. In the sharecropping model given in Stiglitz (RES, 1974), the principal is now a landlord and the agent is the landlord's tenant. By exerting an effort e in $\{0,1\}$, the tenant increases (decreases) the probability $\pi(e)$ (resp. $1-\pi(e)$) that a large $\bar{q}$ (resp. small $\underline{q}$) quantity of an agricultural product is produced. The price of this good is normalized to one so that the principal's stochastic return on the activity is also $\bar{q}$ or $\underline{q}$, depending on the state of nature.

It is often the case that peasants in developing countries are subject to strong financial constraints. To model such a setting we assume that the agent is risk neutral and protected by limited liability. When he wants to induce effort, the principal's optimal contract must solve

$$\max_{\{(\underline{t},\bar{t})\}} \pi_1(\bar{q}-\bar{t})+(1-\pi_1)(\underline{q}-\underline{t}) \tag{8.52}$$

subject to

$$\pi_1\bar{t} + (1-\pi_1)\underline{t} - \psi \geqq \pi_0\bar{t} + (1-\pi_0)\underline{t}, \tag{8.53}$$

$$\pi_1\bar{t} + (1-\pi_1)\underline{t} - \psi \geqq 0, \tag{8.54}$$

$$\underline{t} \geqq 0. \tag{8.55}$$

The optimal contract therefore satisfies $\underline{t}^{SB} = 0$ and $\bar{t}^{SB} = \frac{\psi}{\Delta\pi}$. This is again akin to an efficiency wage. The expected utilities obtained respectively by the principal and the agent are given by

$$EV^{SB} = \pi_1\bar{q} + (1-\pi_1)\underline{q} - \frac{\pi_1\psi}{\Delta\pi}. \tag{8.56}$$

and

$$EU^{SB} = \frac{\pi_0\psi}{\Delta\pi}. \tag{8.57}$$

The flexible second-best contract described above has sometimes been criticized as not corresponding to the contractual arrangements observed in most agrarian economies. Contracts often take the form of simple linear schedules linking the tenant's production to his compensation. As an exercise, let us now analyze a simple linear sharing rule between the landlord and his tenant, with the landlord offering the agent a fixed share α of the realized production. Such a sharing rule automatically satisfies the agent's limited liability constraint, which can therefore be omitted in what follows. Formally, the optimal linear rule inducing effort must solve

$$\max_{\alpha}(1-\alpha)(\pi_1\bar{q} + (1-\pi_1)\underline{q}) \tag{8.58}$$

subject to

$$\alpha(\pi_1\bar{q} + (1-\pi_1)\underline{q}) - \psi \geqq \alpha(\pi_0\bar{q} + (1-\pi_0)\underline{q}), \tag{8.59}$$

$$\alpha(\pi_1\bar{q} + (1-\pi_1)\underline{q}) - \psi \geqq 0 \tag{8.60}$$

Obviously, only (8.59) is binding at the optimum. One finds the optimal linear sharing rule to be

$$\alpha^{SB} = \frac{\psi}{\Delta\pi\Delta q}. \tag{8.61}$$

Note that $\alpha^{SB} < 1$ because, for the agricultural activity to be a valuable venture in the first-best world, we must have $\Delta\pi\Delta q > \psi$. Hence, the return on the agricultural activity is shared between the principal and the agent, with high-powered incentives (α

close to one) being provided when the disutility of effort ψ is large or when the principal's gain from an increase of effort $\Delta\pi\Delta q$ is small.

This sharing rule also yields the following expected utilities to the principal and the agent, respectively

$$EV_\alpha = \pi_1\bar{q} + (1-\pi_1)\underline{q} - \left(\frac{\pi_1\bar{q} + (1-\pi_1)\underline{q}}{\Delta q}\right)\frac{\psi}{\Delta\pi} \tag{8.62}$$

and

$$EU_\alpha = \left(\frac{\pi_1\bar{q} + (1-\pi_1)\underline{q}}{\Delta q}\right)\frac{\psi}{\Delta\pi}. \tag{8.63}$$

Comparing (8.56) and (8.62) on the one hand and (8.57) and (8.63) on the other hand, we observe that the constant sharing rule benefits the agent but not the principal. A linear contract is less powerful than the optimal second-best contract. The former contract is an inefficient way to extract rent from the agent even if it still provides sufficient incentives to exert effort. Indeed, with a linear sharing rule, the agent always benefits from a positive return on his production, even in the worst state of nature. This positive return yields to the agent more than what is requested by the optimal second-best contract in the worst state of nature, namely zero. Punishing the agent for a bad performance is thus found to be rather difficult with a linear sharing rule.

A linear sharing rule allows the agent to keep some strictly positive rent EU_α. If the space of available contracts is extended to allow for fixed fees β, the principal can nevertheless bring the agent down to the level of his outside opportunity by setting a fixed fee β^{SB} equal to $\left(\frac{\pi_1\bar{q}+(1-\pi_1)\underline{q}}{\Delta q}\right)\frac{\psi}{\Delta\pi}$.

8.7.3 Wholesale Contracts

Let us now consider a manufacturer-retailer relationship studied in Laffont and Tirole (1993). The manufacturer supplies at constant marginal cost c an intermediate good to the risk-averse retailer, who sells this good on a final market. Demand on this market is high (resp. low) $\bar{D}(p)$ (resp. $\underline{D}(p)$) with probability $\pi(e)$ where, again, e is in $\{0,1\}$ and p denotes the price for the final good. Effort e is exerted by the retailer, who can increase the probability that demand is high if after-sales services are efficiently performed. The wholesale contract consists of a retail price maintenance agreement specifying the prices $\bar{p}$ and $\underline{p}$ on the final market with a sharing of the profits, namely $\{(\underline{t},\underline{p});(\bar{t},\bar{p})\}$. When he wants to induce effort, the optimal contract offered by the manufacturer solves the following problem:

$$\max_{\{(\underline{t},\underline{p});(\bar{t},\bar{p})\}} \pi_1((\bar{p}-c)\bar{D}(\bar{p})-\bar{t})+(1-\pi_1)((\underline{p}-c)\underline{D}(\underline{p})-\underline{t}) \tag{8.64}$$

subject to (8.3) and (8.4).

The solution to this problem is obtained by appending the following expressions of the retail prices to the transfers given in (8.33) and (8.34): $\bar{p}^* + \frac{\bar{D}(\bar{p}^*)}{D'(\bar{p}^*)} = c$, and $\underline{p}^* + \frac{\underline{D}(\underline{p}^*)}{\underline{D}'(\underline{p}^*)} = c$. Note that these prices are the same as those that would be chosen under complete information. The pricing rule is not affected by the incentive problem.

8.7.4 Financial Contracts

Moral hazard is an important issue in financial markets. In Holmstrom and Tirole (AER, 1994), it is assumed that a risk-averse entrepreneur wants to start a project that requires an initial investment worth an amount I. The entrepreneur has no cash of his own and must raise money from a bank or any other financial intermediary. The return on the project is random and equal to $\bar{V}$ (resp. $\underline{V}$) with probability $\pi(e)$ (resp. $1-\pi(e)$), where the effort exerted by the entrepreneur e belongs to $\{0,1\}$. We denote the spread of profits by $\Delta V = \bar{V} - \underline{V} > 0$. The financial contract consists of repayments $\{(\bar{z},\underline{z})\}$, depending upon whether the project is successful or not.

To induce effort from the borrower, the risk-neutral lender's program is written as

$$\max_{\{(\underline{z},\bar{z})\}} \pi_1\bar{z} + (1-\pi_1)\underline{z} - I \tag{8.65}$$

subject to

$$\begin{aligned} & \pi_1 u(\bar{V}-\bar{z}) + (1-\pi_1)u(\underline{V}-\underline{z}) - \psi \\ \geqq \;& \pi_0 u(\bar{V}-\bar{z}) + (1-\pi_0)u(\underline{V}-\underline{z}), \end{aligned} \tag{8.66}$$

$$\pi_1 u(\bar{V}-\bar{z}) + (1-\pi_1)u(\underline{V}-\underline{z}) - \psi \geqq 0. \tag{8.67}$$

Note that the project is a valuable venture if it provides the bank with a positive expected profit.

With the change of variables, $\bar{t} = \bar{V} - \bar{z}$ and $\underline{t} = \underline{V} - \underline{z}$, the principal's program takes its usual form. This change of variables also highlights the fact that everything happens as if the lender was benefitting directly from the returns of the project, and then paying the agent only a fraction of the returns in the different states of nature.

Let us define the second-best cost of implementing a positive effort C^{SB}, and let us assume that $\Delta\pi\Delta V \geqq C^{SB}$, so that the lender wants to induce a positive effort level even in a second-best environment. The lender's expected profit is worth

$$V_1 = \pi_1\bar{V} + (1-\pi_1)\underline{V} - C^{SB} - I. \quad (8.68)$$

Let us now parameterize projects according to the size of the investment I. Only the projects with positive value $V_1 > 0$ will be financed. This requires the investment to be low enough, and typically we must have

$$I < I^{SB} = \pi_1\bar{V} + (1-\pi_1)\underline{V} - C^{SB}. \quad (8.69)$$

Under complete information and no moral hazard, the project would instead be financed as soon as

$$I < I^* = \pi_1\bar{V} + (1-\pi_1)\underline{V} \quad (8.70)$$

For intermediary values of the investment. i.e., for I in $[I^{SB}, I^*]$, moral hazard implies that some projects are financed under complete information but no longer under moral hazard. This is akin to some form of credit rationing.

Finally, note that the optimal financial contract offered to the risk-averse and cashless entrepreneur does not satisfy the limited liability constraint $\underline{t} \geqq 0$. Indeed, we have $\underline{t}^{SB} = h\left(\psi - \frac{\pi_1\psi}{\Delta\pi}\right) < 0$. To be induced to make an effort, the agent must bear some risk, which implies a negative payoff in the bad state of nature. Adding the limited liability constraint, the optimal contract would instead entail $\underline{t}^{LL} = 0$ and $\bar{t}^{LL} = h\left(\frac{\psi}{\Delta\pi}\right)$. Interestingly, this contract has sometimes been interpreted in the corporate finance literature as a debt contract, with no money being left to the borrower in the bad state of nature and the residual being pocketed by the lender in the good state of nature.

Finally, note that

$$\begin{aligned}\bar{t}^{LL} - \underline{t}^{LL} &= h\left(\frac{\psi}{\Delta\pi}\right) < \bar{t}^{SB} - \underline{t}^{SB} = h\left(\psi + (1-\pi_1)\frac{\psi}{\Delta\pi}\right) \\ &\quad - h\left(\psi - \frac{\pi_1\psi}{\Delta\pi}\right),\end{aligned} \quad (8.71)$$

since $h(\cdot)$ is strictly convex and $h(0) = 0$. This inequality shows that the debt contract has less incentive power than the optimal incentive contract. Indeed, it becomes harder to spread the agent's payments between both states of nature to induce effort if the agent is protected by limited liability by the agent, who is interested only in his payoff in the high state of nature, only rewards are attractive.

8.8 A Continuum of Performances

Let us now assume that the level of performance $\tilde{q}$ is drawn from a continuous distribution with a cumulative function $F(\cdot|e)$ on the support $[\underline{q}, \bar{q}]$. This distribution is conditional on the agent's level of effort, which still takes two possible values e in $\{0, 1\}$. We denote by $f(\cdot|e)$ the density corresponding to the above distributions. A contract $t(q)$ inducing a positive effort in this context must satisfy the incentive constraint

$$\int_{\underline{q}}^{\bar{q}} u(t(q))f(q|1)dq - \psi \geqq \int_{\underline{q}}^{\bar{q}} u(t(q))f(q|0)dq, \tag{8.72}$$

and the participation constraint

$$\int_{\underline{q}}^{\bar{q}} u(t(q))f(q|1)dq - \psi \geqq 0. \tag{8.73}$$

The risk-neutral principal problem is thus written as

$$\max_{\{t(q)\}} \int_{\underline{q}}^{\bar{q}} (S(q) - t(q))f(q|1)dq, \tag{8.74}$$

subject to (8.72) and (8.73).

Denoting the multipliers of (8.72) and (8.73) by λ and μ, respectively, the Lagrangian is written as

$$L(q,t) = (S(q) - t)f(q|1) + \lambda(u(t)(f(q|1) - f(q|0)) - \psi) + \mu(u(t)f(q|1) - \psi).$$

Optimizing pointwise with respect to t yields

$$\frac{1}{u'(t^{SB}(q))} = \mu + \lambda\left(\frac{f(q|1) - f(q|0)}{f(q|1)}\right). \tag{8.75}$$

Multiplying (8.75) by $f_1(q)$ and taking expectations, we obtain, as in the main text,

$$\mu = E_{\tilde{q}}\left(\frac{1}{u'(t^{SB}(\tilde{q}))}\right) > 0, \tag{8.76}$$

where $E_{\tilde{q}}(\cdot)$ is the expectation operator with respect to the probability distribution of output induced by an effort e^{SB}. Finally, using this expression of μ, inserting it into (8.75), and multiplying it by $f(q|1)u(t^{SB}(q))$, we obtain

$$\begin{aligned} & \lambda(f(q|1) - f(q|0))u(t^{SB}(q)) \\ = \ & f(q|1)u(t^{SB}(q))\left(\frac{1}{u'(t^{SB}(q))} - E_{\tilde{q}}\left(\frac{1}{u'(t^{SB}(q))}\right)\right). \end{aligned} \tag{8.77}$$

Integrating over $[\underline{q}, \tilde{q}]$ and taking into account the slackness condition $\lambda(\int_{\underline{q}}^{\bar{q}}(f(q|1) - f(q|0))u(t^{SB}(q))dq - \psi) = 0$ yields $\lambda\psi = \text{cov}(u(t^{SB}(\tilde{q})), \frac{1}{u'(t^{SB}(\tilde{q}))}) \geqq 0$.

Hence, $\lambda \geqq 0$ because $u(\cdot)$ and $u'(\cdot)$ vary in opposite directions. Also, $\lambda = 0$ only if $t^{SB}(q)$ is a constant, but in this case the incentive constraint is necessarily violated. As a result, we have $\lambda > 0$. Finally, $t^{SB}(\pi)$ is monotonically increasing in π when the monotone likelihood property $\frac{d}{dq}\left(\frac{f(q|1) - f*(q|0)}{f(q|1)}\right) \geqq 0$ is satisfied.

8.9 Further Extension

We have stressed the various conflicts that may appear in a moral hazard environment. The analysis of these conflicts, under both limited liability and risk aversion, was made easy because of our focus on a simple 2×2 environment with a binary effort and two levels of performance. The simple interaction between a single incentive constraint with either a limited liability constraint or a participation constraint was quite straightforward.

When one moves away from the 2×2 model, the analysis becomes much harder, and characterizing the optimal incentive contract is a difficult task. Examples of such complex contracting environment are abound. Effort may no longer be binary but, instead, may be better characterized as a continuous variable. A manager may no longer choose between working or not working on a project but may be able to fine-tune the exact effort spent on this project. Even worse, the agent's actions may no longer be summarized by a one-dimensional parameter but may be better described by a whole array of control variables that are technologically linked. For instance, the manager of a firm may have to choose how to allocate his effort between productive activities and monitoring his peers and other workers.

Nevertheless, one can extend the standard model to the cases where the agent can perform more than two and possibly a continuum of levels of effort, to the case with a multitask model, the case where the agent's utility function is no longer separable between consumption and effort. One can also analyze the trade-off between efficiency and redistribution in a moral hazard context. For detailed discussion, see Chapter 5 of Laffont and Martimort (2002).

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Chapter 9

General Mechanism Design

9.1 Introduction

In the previous chapters on the principal-agent theory, we have introduced basic models to explain the core of the principal-agent theory with complete contracts. It highlights the various trade-offs between allocative efficiency and the distribution of information rents. Since the model involves only one agent, the design of the principal's optimal contract has reduced to a constrained optimization problem without having to appeal to sophisticated game theory concepts.

In this chapter, we will introduce some of basic results and insights of the mechanism design in general, and implementation theory in particular for situations where there is one principal (also called the designer) and several agents. In such a case, asymmetric information may not only affect the relationship between the principal and each of his agents, but it may also plague the relationships between agents. To describe the strategic interaction between agents and the principal, the game theoretic reasoning is thus used to model social institutions as varied voting systems, auctions, bargaining protocols, and methods for deciding on public projects.

Incentive problems arise when the social planner cannot distinguish between things that are indeed different so that free-ride problem many appear. A free rider can improve his welfare by not telling the truth about his own un-observable characteristic. Like the principal-agent model, a basic insight of the incentive mechanism with more than one agent is that incentive constraints should be considered coequally with resource constraints. One of the most fundamental contributions of the mechanism theory has been shown that the free-rider problem may or may not occur, depending on the kind of game (mechanism) that agents play and other game theoretical solution concepts. A theme that comes out of the literature is the difficulty of finding mechanisms compatible with individual incentives that simultaneously results in a desired social goal.

Examples of incentive mechanism design that takes strategic interactions among agents exist for a long time. An early example is the Biblical story of the famous judgement of Solomon for determining who is the real mother of a baby. Two women came before the King, disputing who was the mother of a child. The King's solution used a method of threatening to cut the lively baby in two and give half to each. One women was willing to give up the child, but another women agreed to cut in two. The King then made his judgement and decision: The first woman is the mother, do not kill the child and give the him to the first woman. Another example of incentive mechanism design is how to cut a pie and divide equally among all participants.

The first major development was in the work of Gibbard-Hurwicz-Satterthwaite in 1970s. When information is private, the appropriate equilibrium concept is dominant strategies. These incentives adopt the form of incentive compatibility constraints where for each agent to tell truth about their characteristics must be dominant. The fundamental conclusion of Gibbard-Hurwicz-Satterthwaite's impossibility theorem is that we have to have a trade-off between the truth-telling and Pareto efficiency (or the first best outcomes in general). Of course, if one is willing to give up Pareto efficiency, we can have a truth-telling mechanism, such as Groves-Clark mechanism. In many cases, one can ignore the first-best or Pareto efficiency, and so one can expect the truth- telling behavior.

On the other hand, we could give up the truth-telling requirement, and want to reach Pareto efficient outcomes. When the information about the characteristics of the agents is shared by individuals but not by the designer, then the relevant equilibrium concept is the Nash equilibrium. In this situation, one can gives up the truth-telling, and uses a general message space. One may design a mechanism that Nash implements Pareto efficient allocations.

We will introduce these results and such trade-offs. We will also briefly introduce the incomplete information case in which agents do not know each other's characteristics, and we need to consider Bayesian incentive compatible mechanism.

9.2 Basic Settings

Theoretical framework of the incentive mechanism design consists of five components: (1) economic environments (fundamentals of economy); (2) social choice goal to be reached; (3) economic mechanism that specifies the rules of game; (4) description of solution concept on individuals' self-interested behavior, and (5) implementation of a social choice goal (incentive-compatibility of personal interests and the social goal at equilibrium).

9.2.1 Economic Environments

$e_i = (Z_i, w_i, \succcurlyeq_i, Y_i)$: economic characteristic of agent i which consists of outcome space, initial endowment if any, preference relation, and the production set if agent i is also a producer;

$e = (e_1, \ldots, e_n)$: an economy;

E: The set of all priori admissible economic environments.

$U = U_1 \times \ldots \times U_n$: The set of all admissible utility functions.

Remark 9.2.1 Here, E is a general expression of economic environments. However, depending on the situations facing the designer, the set of admissible economic environments under consideration sometimes may be just given by $E = U$, or by the set of all possible initial endowments, or production sets.

The designer is assumed that he does not know the individuals' economic characteristics. The individuals may or may not know the characteristics of the others. If they know, it is called the complete information case, otherwise it is called the incomplete information case.

9.2.2 Social Goal

Given economic environments, each agent participates economic activities, makes decisions, receives benefits and pays costs on economic activities. Let

$Z = Z_1 \times \ldots \times Z_n$: the outcome space (For example, $Z = X \times Y$).

$A \subseteq Z$: the feasible set.

$F : E \to\to A$: the social goal or called social choice correspondence in which $F(e)$ is the set of socially desired outcomes at the economy under some criterion of social optimality.

Examples of Social Choice Correspondences:

$P(e)$: the set of Pareto efficient allocations.

$I(e)$: the set of individual rational allocations.

$W(e)$: the set of Walrasian allocations.

$L(e)$: the set of Lindahl allocations.

$FA(e)$: the set of fare allocations.

When F becomes a single-valued function, denoted by f, it is called a social choice function.

Examples of Social Choice Functions:

Solomon's goal.

Majority voting rule.

9.2.3 Economic Mechanism

Since the designer lacks the information about individuals' economic characteristics, he needs to design an appropriate incentive mechanism (rules of game) to coordinate the personal interests and the social goal, i.e., under the mechanism, all individuals have incentives to choose actions which result in socially optimal outcomes when they pursue their personal interests. To do so, the designer informs how the information he collected from individuals is used to determine outcomes, that is, he first tells the rules of games. He then uses the information or actions of agents and the rules of game to determine outcomes of individuals. Thus, a mechanism consists of a message space and an outcome function. Let

M_i : the message space of agent i.

$M = M_1 \times \ldots \times M_n$: the message space in which communications take place.

$m_i \in M_i$: a message reported by agent i.

$m = (m_1, \ldots, m_n) \in M$: a profile of messages.

$h : M \to Z$: outcome function that translates messages into outcomes.

$\Gamma =< M, h >$: a mechanism

That is, a mechanism consists of a message space and an outcome function.

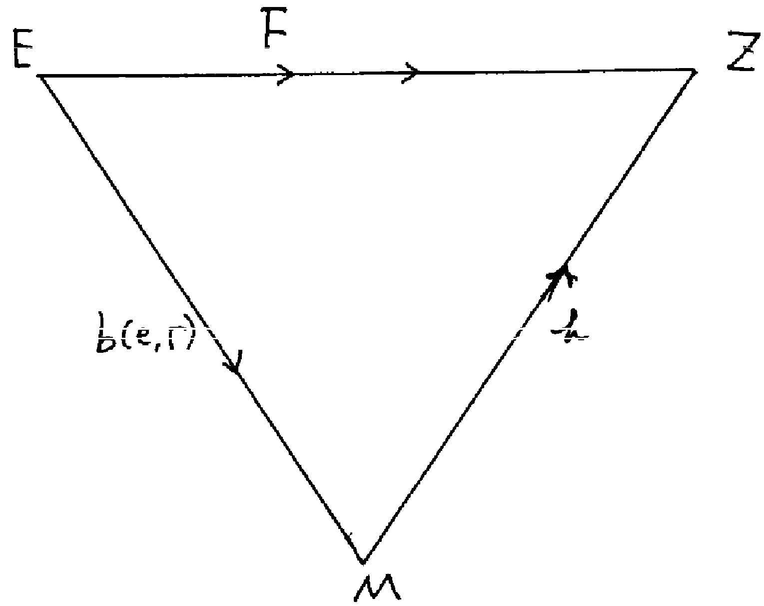

Figure 9.1: Diagrammatic Illustration of Mechanism design Problem.

Remark 9.2.2 A mechanism is often also referred to as a game form. The terminology of game form distinguishes it from a game in game theory, as the consequence of a profile of message is an outcome rather than a vector of utility payoffs. However, once the preference of the individuals are specified, then a game form or mechanism induces a conventional game. Since the preferences of individuals in the mechanism design setting vary, this distinction between mechanisms and games is critical.

Remark 9.2.3 In the implementation (incentive mechanism design) literature, one requires a mechanism be incentive compatible in the sense that personal interests are consistent with desired socially optimal outcomes even when individual agents are self-interested in their personal goals without paying much attention to the size of message. In the realization literature originated by Hurwicz (1972, 1986b), a sub-field of the mechanism literature, one also concerns the size of message space of a mechanism, and tries to find economic system to have small operation cost. The smaller a message space of a mechanism, the lower (transition) cost of operating the mechanism. For the neoclassical economies, it has been shown that competitive market economy system is the unique most efficient system that results in Pareto efficient and individually rational allocations (cf, Mount and Reiter (1974), Walker (1977), Osana (1978), Hurwicz (1986b), Jordan (1982), Tian (2002d)).

9.2.4 Solution Concept of Self-Interested Behavior

In economics, a basic assumption is that individuals are self-interested in the sense that they pursue their personal interests. Unless they can be better off, they in general does not care about social interests. As a result, different economic environments and different rules of game will lead to different reactions of individuals, and thus each individual agent's strategy on reaction will depend on his self-interested behavior which in turn depends on the economic environments and the mechanism.

Let $b(e, \Gamma)$ be the set of equilibrium strategies that describes the self-interested behavior of individuals. Examples of such equilibrium solution concepts include Nash equilibrium, dominant strategy, Bayesian Nash equilibrium, etc.

Thus, given E, M, h, and b, the resulting equilibrium outcome is the composite function of the rules of game and the equilibrium strategy, i.e., $h(b(e, \Gamma))$.

9.2.5 Implementation and Incentive Compatibility

In which sense can we see individuals's personal interests do not have conflicts with a social interest? We will call such problem as implementation problem. The purpose of an incentive mechanism design is to implement some desired socially optimal outcomes. Given a mechanism Γ and equilibrium behavior assumption $b(e, \Gamma)$, the implementation problem of a social choice rule F studies the relationship of the intersection state of $F(e)$ and $h(b(e, \Gamma))$. Thus, we have the following various concepts on implementation and incentive compatibility of F.

A Mechanism $< M, h >$ is said to

(i) fully implement a social choice correspondence F in equilibrium strategy $b(e, \Gamma)$ on E if for every $e \in E$

(a) $b(e, \Gamma) \neq \emptyset$ (equilibrium solution exists),

(b) $h(b(e, \Gamma)) = F(e)$ (personal interests are fully consistent with social goals);

(ii) implement a social choice correspondence F in equilibrium strategy $b(e, \Gamma)$ on E if for every $e \in E$

(a) $b(e, \Gamma) \neq \emptyset$,

(b) $h(b(e, \Gamma)) \subseteq F(e)$;

(iii) weakly implement a social choice correspondence F in equilibrium strategy $b(e, \Gamma)$ on E if for every $e \in E$

(a) $b(e, \Gamma) \neq \emptyset$,

(b) $h(b(e, \Gamma)) \cap F(e) \neq \emptyset$.

A Mechanism $< M, h >$ is said to be $b(e, \Gamma)$ *incentive-compatible with a social choice correspondence* F in $b(e, \Gamma)$-equilibrium if it (fully or weakly) implements F in $b(e, \Gamma)$-equilibrium.

Note that we did not give a specific solution concept so far when we define the implementability and incentive-compatibility. As shown in the following, whether or not a social choice correspondence is implementable will depend on the assumption on the solution concept of self-interested behavior. When information is complete, the solution concept can be dominant equilibrium, Nash equilibrium, strong Nash equilibrium, subgame perfect Nash equilibrium, undominanted equilibrium, etc. For incomplete information, equilibrium strategy can be Bayesian Nash equilibrium, undominated Bayesian Nash equilibrium, etc.

9.3 Examples

Before we discuss some basic results in the mechanism theory, we first give some economic environments which show that one needs to design a mechanism to solve the incentive compatible problems.

Example 9.3.1 (A Public Project) A society is deciding on whether or not to build a public project at a cost c. The cost of the pubic project is to be divided equally.

The outcome space is then $Y = \{0, 1\}$, where 0 represents not building the project and 1 represents building the project. Individual i's value from use of this project is r_i. In this case, the net value of individual i is 0 from not having the project built and $v_i \equiv r_i - \frac{c}{n}$ from having a project built. Thus agent i's valuation function can be represented as

$$v_i(y, v_i) = yr_i - y\frac{c}{n} = yv_i.$$

Example 9.3.2 (Continuous Public Goods Setting) In the above example, the public good could only take two values, and there is no scale problem. But, in many case, the level of public goods depends on the collection of the contribution or tax. Now let $y \in R_+$ denote the scale of the public project and $c(y)$ denote the cost of producing y. Thus, the outcome space is $Z = R_+ \times R^n$, and the feasible set is $A = \{(y, z_1(y), \ldots, z_n(y)) \in R_+ \times R^n : \sum_{i \in N} z_i(y) = c(y)\}$, where $z_i(y)$ is the share of agent i for producing the public goods y. The benefit of i for building y is $r_i(y)$ with $r_i(0) = 0$. Thus, the net benefit of not building the project is equal to 0, the net benefit of building the project is $r_i(y) - z_i(y)$. The valuation function of agent i can be written as

$$v_i(y) = r_i(y) - z_i(y).$$

Example 9.3.3 (Allocating an Indivisible Private Good) An indivisible good is to be allocated to one member of society. For instance, the rights to an exclusive license are to be allocated or an enterprise is to be privatized. In this case, the outcome space is $Z = \{y \in \{0, 1\}^n : \sum_{i=1}^n y_i = 1\}$,where $y_i = 1$ means individual i obtains the object, $y_i = 0$ represents the individual does not get the object. If individual i gets the object, the net value benefitted from the object is v_i. If he does not get the object, his net value is 0. Thus, agent i's valuation function is

$$v_i(y) = v_i y_i.$$

Note that we can regard y as n-dimensional vector of public goods since $v_i(y) = v_i y_i = vy$, where $v = (v_1, \ldots, v_n)$.

From these examples, a socially optimal decision clearly depends on the individuals' true valuation function $v_i(\cdot)$. For instance, we have shown previously that a public project is produced if and only if the total values of all individuals is greater than it total cost, i.e., if $\sum_{i \in N} r_i > c$, then $y = 1$, and if $\sum_{i \in N} r_i < c$, then $y = 0$.

Let V_i be the set of all valuation functions v_i, let $V = \prod_{i \in N} V_i$, let $h : V \to Z$ is a decision rule. Then h is said to be efficient if and only if:

$$\sum_{i \in N} v_i(h(v_i)) \geqq \sum_{i \in N} v_i(h(v_i')) \quad \forall v' \in V.$$

9.4 Dominant Strategy and Truthful Revelation Mechanism

The strongest solution concept of describing self-interested behavior is dominant strategy. The dominant strategy identifies situations in which the strategy chosen by each individual is the best, regardless of choices of the others. An axiom in game theory is that agents will use it as long as a dominant strategy exists.

For $e \in E$, a mechanism $\Gamma =< M, h >$ is said to have a dominant strategy equilibrium m^* if for all i

$$h_i(m_i^*, m_{-i}) \succcurlyeq_i h_i(m_i, m_{-i}) \text{ for all } m \in M. \tag{9.1}$$

Denoted by $D(e, \Gamma)$ the set of dominant strategy equilibria for $\Gamma =< M, h >$ and $e \in E$.

Under the assumption of dominant strategy, since each agent's optimal choice does not depend on the choices of the others and does not need to know characteristics of the others, the required information is least when an individual makes decisions. Thus, if it exists, it is an ideal situation.

When the solution concept is given by dominant strategy equilibrium, i.e., $b(e, \Gamma) = D(e, \Gamma)$, a mechanism $\Gamma =< M, h >$ implements a social choice correspondence F in dominant equilibrium strategy on E if for every $e \in E$,

(a) $D(e, \Gamma) \neq \emptyset$;

(b) $h(D(e, \Gamma)) \subset F(e)$.

The above definitions have applied to general (indirect) mechanisms, there is, however, a particular class of game forms which have a natural appeal and have received much attention in the literature. These are called direct or revelation mechanisms, in which the message space M_i for each agent i is the set of possible characteristics E_i. In effect, each agent reports a possible characteristic but not necessarily his true one.

A mechanism $\Gamma =< M, h >$ is said to be *a revelation or direct mechanism* if $M = E$.

Example 9.4.1 Groves mechanism is a revelation mechanism.

The most appealing revelation mechanisms are those in which truthful reporting of characteristics always turns out to be an equilibrium. It is the absence of such a mechanism which has been called the "free-rider" problem in the theory of public goods. Perhaps the most appealing revelation mechanisms of all are those for which each agent has truth as a dominant strategy.

A revelation mechanism $< E, h >$ is said to implements a social choice correspondence F truthfully in $b(e, \Gamma)$ on E if for every $e \in E$,

(a) $e \in b(e, \Gamma)$;

(b) $h(e) \subset F(e)$.

Although the message space of a mechanism can be arbitrary, the following Revelation Principle tells us that one only needs to use the so-called revelation mechanism in which the message space consists solely of the set of individuals' characteristics, and it is unnecessary to seek more complicated mechanisms. Thus, it will significantly reduce the complicity of constructing a mechanism.

Theorem 9.4.1 (Revelation Principle) *Suppose a mechanism $< M, h >$ implements a social choice rule F in dominant strategy. Then there is a revelation mechanism $< E, g >$ which implements F truthfully in dominant strategy.*

Proof. Let d be a selection of dominant strategy correspondence of the mechanism $< M, h >$, i.e., for every $e \in E$, $m^* = d(e) \in D(e, \Gamma)$. Since $\Gamma = \langle M, h \rangle$ implements social choice rule F, such a selection exists by the implementation of F. Since the strategy of each agent is independent of the strategies of the others, each agent i's dominant strategy can be expressed as $m_i^* = d_i(e_i)$.

Define the revelation mechanism $< E, g >$ by $g(e) \equiv h(d(e))$ for each $e \in E$. We first show that the truth-telling is always a dominant strategy equilibrium of the revelation mechanism $\langle E, g \rangle$. Suppose not. Then, there exists a e' and an agent i such that

$$u_i[g(e'_i, e'_{-i})] > u_i[g(e_i, e'_{-i})].$$

However, since $g = h \circ d$, we have

$$u_i[h(d(e'_i), d(e'_{-i})] > u_i[h(d(e_i), d(e'_{-i})],$$

which contradicts the fact that $m^*_i = d_i(e_i)$ is a dominant strategy equilibrium. This is because, when the true economic environment is (e_i, e'_{-i}), agent i has an incentive not to report $m^*_i = d_i(e_i)$ truthfully, but have an incentive to report $m'_i = d_i(e'_i)$, a contradiction.

Finally, since $m^* = d(e) \in D(e, \Gamma)$ and $< M, h >$ implements a social choice rule F in dominant strategy, we have $g(e) = h(d(e)) = h(m^*) \in F(e)$. Hence, the revelation mechanism implements F truthfully in dominant strategy. The proof is completed.

Thus, by the Revelation Principle, we know that, if truthful implementation rather than implementation is all that we require, we need never consider general mechanisms. In the literature, if a revelation mechanism $< E, h >$ truthfully implements a social choice rule F in dominant strategy, the mechanism Γ is said to be *strongly incentive-compatible* with a social choice correspondence F. In particular, when F becomes a single-valued function f, $< E, f >$ can be regarded as a revelation mechanism. Thus, if a mechanism $< M, h >$ implements f in dominant strategy, then the revelation mechanism $< E, f >$ is incentive compatible in dominant strategy, or called strongly incentive compatible.

Remark 9.4.1 Notice that the Revelation Principle may be valid only for weak implementation. The Revelation Principle specifies a correspondence between a dominant strategy equilibrium of the original mechanism $< M, h >$ and the true profile of characteristics as a dominant strategy equilibrium, and it does not require the revelation mechanism has a unique dominant equilibrium so that the revelation mechanism $< E, g >$ may also exist non-truthful strategy equilibrium that does not corresponds to any equilibrium. Thus, in moving from the general (indirect) dominant strategy mechanisms to direct ones, one may introduce dominant strategies which are not truthful. More troubling, these additional strategies may create a situation where the indirect mechanism is an implantation of a given F, while the direct revelation mechanism is not. Thus, even if a mechanism implements a social choice function, the corresponding revelation mechanism $< E, g >$ may only weakly implement, but not implement F.

9.5 Gibbard-Satterthwaite Impossibility Theorem

The Revelation Principle is very useful to find a dominant strategy mechanism. If one hopes a social choice goal f can be (weakly) implemented in dominant strategy, one only needs to show the revelation mechanism $< E, f >$ is strongly incentive compatible. However, the Gibbard-Satterthwaite impossibility theorem in Chapter 4 tells us that, if the domain of economic environments is unrestricted, such a mechanism does not exist unless it is a dictatorial mechanism. From the angle of the mechanism design, we state this theorem repeatly here.

Definition 9.5.1 A social choice function is dictatorial if there exists an agent whose optimal choice is the social optimal.

Now we state the Gibbard-Satterthwaite Theorem without the proof that is very complicated. A proof can be found, say, in Salanié's book (2000): *Microeconomics of Market Failures.*

Theorem 9.5.1 (Gibbard-Satterthwaite Theorem) *If X has at least 3 alternatives, a social choice function which is strongly incentive compatible and defined on a unrestricted domain is dictatorial.*

9.6 Hurwicz Impossibility Theorem

The Gibbard-Satterthwaite impossibility theorem is a very negative result. This result is very similar to Arrow's impossibility result. However, as we will show, when the admissible set of economic environments is restricted, the result may be positive as the Groves mechanism defined on quasi-linear utility functions. Unfortunately, the following Hurwicz's impossibility theorem shows the Pareto efficiency and the truthful revelation is fundamentally inconsistent even for the class of neoclassical economic environments.

Theorem 9.6.1 (Hurwicz Impossibility Theorem, 1972) *For the neoclassical private goods economies, any mechanism $< M, h >$ that yields Pareto efficient and individually rational allocations is not strongly individually incentive compatible. (Truth-telling about their preferences is not Nash Equilibrium).*

Proof: By the Revelation Principle, we only need to consider any revelation mechanism that cannot implement Pareto efficient and individually rational allocations in dominant equilibrium for a particular pure exchange economy.

Consider a private goods economy with two agents ($n = 2$) and two goods ($L = 2$),

$$\begin{aligned} w_1 &= (0, 2), w_2 = (2, 0) \\ u_i(x, y) &= \begin{cases} 3x_i + y_i & if \;\; x_i \leqq y_i \\ x_i + 3y_i & if \;\; x_i > y_i \end{cases} \end{aligned}$$

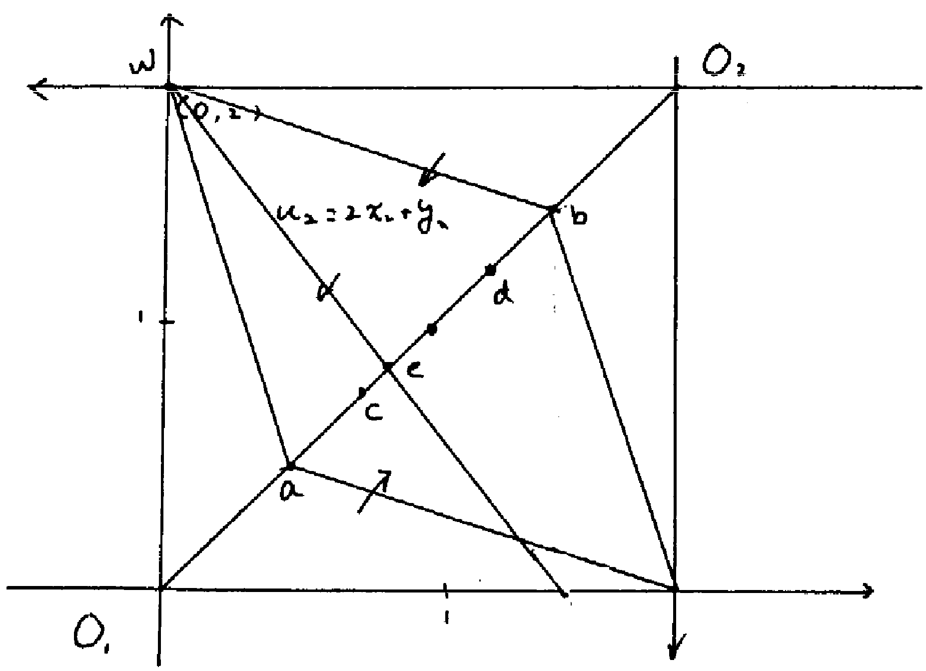

Figure 9.2: An illustration of the proof of Hurwicz's impossibility theorem.

Thus, feasible allocations are given by

$$\begin{aligned} A &= \{[(x_1, y_1), (x_2, y_2)] \in R^4_+ : \\ & x_1 + x_2 = 2 \\ & y_1 + y_2 = 2\} \end{aligned}$$

U_i is the set of all neoclassical utility functions, i.e. they are continuous and quasi-concave, which agent i can report to the designer. Thus, the true utility function $\mathring{u}_i \in U_i$. Then,

$$\begin{aligned} U &= U_1 \times U_2 \\ h &: U \to A \end{aligned}$$

Note that, if the true utility function profile $\mathring{u}_i$ is a Nash Equilibrium, it satisfies

$$\mathring{u}_i(h_i(\mathring{u}_i, \mathring{u}_{-i})) \geqq \mathring{u}_i(h_i(u_i, \mathring{u}_{-i})) \tag{9.2}$$

We want to show that $\mathring{u}_i$ is not a Nash equilibrium. Note that,

(1) $P(e) = \overline{O_1O_2}$ (contract curve)

(2) $IR(e) \cap P(e) = \overline{ab}$

(3) $h(\mathring{u}_1, \mathring{u}_2) = d \in \overline{ab}$

Now, suppose agent 2 reports his utility function by cheating:

$$u_2(x_2, y_2) = 2x + y \tag{9.3}$$

Then, with u_2, the new set of individually rational and Pareto efficient allocations is given by

$$IR(e) \cap P(e) = \overline{ae} \tag{9.4}$$

Note that any point between a and e is strictly preferred to d by agent 2. Thus, the allocation determined by any mechanism which yields IR and Pareto efficient allocation under $(\mathring{u}_1, u_2)$ is some point, say, the point c in the figure, between the segment of the line determined by a and e. Hence, we have

$$\mathring{u}_2(h_2(\mathring{u}_1, u_2)) > \mathring{u}_2(h_2(\mathring{u}_1, \mathring{u}_2)) \tag{9.5}$$

since $h_2(\mathring{u}_1, u_2) = c \in \overline{ae}$. Similarly, if d is between $\overline{ae}$, then agent 1 has incentive to cheat. Thus, no mechanism that yields Pareto efficient and individually rational allocations is incentive compatible. The proof is completed.

Thus, the Hurwicz's impossibility theorem implies that Pareto efficiency and the truthful revelation about individuals' characteristics are fundamentally incompatible. However, if one is willing to give up Pareto efficiency, say, one only requires the efficient provision of public goods, is it possible to find an incentive compatible mechanism which results in the Pareto efficient provision of a public good and can truthfully reveal individuals' characteristics? The answer is positive. For the class of quasi-linear utility functions, the so-called Groves-Clarke-Vickrey Mechanism can be such a mechanism.

9.7 Groves-Clarke-Vickrey Mechanism

From Chapter 6 on public goods, we have known that public goods economies may present problems by a decentralized resource allocation mechanism because of the free-rider problem. Private provision of a public good generally results in less than an efficient amount of the public good. Voting may result in too much or too little of a public good. Are there any mechanisms that result in the "right" amount of the public good? This is a question of the incentive compatible mechanism design. For simplicity, let us first return to the model of discrete public good.

9.7.1 Groves-Clark Mechanism for Discrete Public Good

Consider a provision problem of a discrete public good. Suppose that the economy has n agents. Let

c: the cost of producing the public project.

r_i: the maximum willingness to pay of i.

g_i: the contribution made by i.

$v_i = r_i - g_i$: the net value of i.

The public project is determined according to

$$y = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} v_i \geqq 0 \\ 0 & \text{otherwise} \end{cases}$$

From the discussion in Chapter 6, it is efficient to produce the public good, $y = 1$, if and only if

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} (r_i - q_i) \geqq 0.$$

Since the maximum willingness to pay for each agent, r_i, is private information and so is the net value v_i, what mechanism one should use to determine if the project is built? One mechanism that we might use is simply to ask each agent to report his or her net value and provide the public good if and only if the sum of the reported value is positive. The problem with such a scheme is that it does not provide right incentives for individual agents to reveal their true willingness-to-pay. Individuals may have incentives to underreport their willingness-to-pay.

Thus, a question is how we can induce each agent to truthfully reveal his true value for the public good. The so-called Groves-Clark mechanism gives such kind of mechanism.

Suppose the utility functions are quasi-linear in net increment in private good, $x_i - w_i$, which have the form:

$$\begin{aligned} \bar{u}_i(x_i, y) &= x_i - w_i + r_i y \\ s.t. \quad x_i + q_i y &= w_i + t_i \end{aligned}$$

where t_i is the transfer to agent i. Then, we have

$$\begin{aligned} u_i(t_i, y) &= t_i + r_i y - g_i y \\ &= t_i + (r_i - g_i) y \\ &= t_i + v_i y. \end{aligned}$$

- Groves Mechanism:

In the Groves mechanism, agents are required to report their net values. Thus the message space of each agent i is $M_i = \Re$. The Groves mechanism is defined as follows:

$\Gamma = (M_1, \ldots, M_n, t_1(\cdot), t_2(\cdot), \ldots, t_n(\cdot), y(\cdot)) \equiv (M, t(\cdot), y(\cdot))$, where

(1) $b_i \in M_i = R$: each agent i reports a "bid" for the public good, i.e., report the net value of agent i which may or may not be his true net value v_i.

(2) The level of the public good is determined by

$$y(b) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} b_i \geqq 0 \\ 0 & \text{otherwise} \end{cases}$$

(3) Each agent i receives a side payment (transfer)

$$t_i = \begin{cases} \sum\limits_{j \neq i} b_j & \text{if } \sum_{i=1}^n b_i \geqq 0 \\ 0 & \text{otherwise} \end{cases} \tag{9.6}$$

Then, the payoff of agent i is given by

$$\phi(b) = \begin{cases} v_i + t_i = v_i + \sum_{j \neq i} b_j & \text{if } \sum_{i=1}^n b_i \geqq 0 \\ 0 & \text{otherwise} \end{cases} \tag{9.7}$$

We want to show that it is optimal for each agent to report the true net value, $b_i = v_i$, regardless of what the other agents report. That is, truth-telling is a dominant strategy equilibrium.

Proposition 9.7.1 *The truth-telling is a dominant strategy under the Groves-Clark mechanism.*

Proof: There are two cases to be considered.

Case 1: $v_i + \sum_{j \neq i} b_j > 0$. Then agent i can ensure the public good is provided by reporting $b_i = v_i$. Indeed, if $b_i = v_i$, then $\sum_{j \neq i} b_j + v_i = \sum_{i=1}^n b_j > 0$ and thus $y = 1$. In this case, $\phi(v_i, b_{-i}) = v_i + \sum_{j \neq i} b_j > 0$.

Case 2: $v_i + \sum_{j \neq i} b_j \leqq 0$. Agent i can ensure that the public good is not provided by reporting $b_i = v_i$ so that $\sum_{i=1}^n b_i \leqq 0$. In this case, $\phi(v_i, b_{-i}) = 0 \geqq v_i + \sum_{j \neq i} b_j$.

Thus, for either cases, agent i has incentives to tell the true value of v_i. Hence, it is optimal for agent i to tell the truth. There is no incentive for agent i to misrepresent his true net value regardless of what other agents do.

The above preference revelation mechanism has a major fault: the total side-payment may be very large. Thus, it is very costly to induce the agents to tell the truth.

Ideally, we would like to have a mechanism where the sum of the side-payment is equal to zero so that the feasibility condition holds, and consequently it results in Pareto efficient allocations, but in general it impossible by Hurwicz's impossibility theorem. However, we could modify the above mechanism by asking each agent to pay a "tax", but not receive payment. Because of this "waster" tax, the allocation of public goods will not be Pareto efficient.

The basic idea of paying a tax is to add an extra amount to agent i's side-payment, $d_i(b_{-i})$ that depends only on what the other agents do.

A General Groves Mechanism: Ask each agent to pay additional tax, $d_i(b_{-i})$.

In this case, the transfer is given by

$$t_i(b) = \begin{cases} \sum_{j\neq t} b_j - d_i(b_{-i}) & \text{if } \sum_{i=1}^n b_i \geqq 0 \\ -d_i(b_{-i}) & \text{if } \sum_{i=1}^n b_i < 0 \end{cases}$$

The payoff to agent i now takes the form:

$$\phi(b) = \begin{cases} v_i + t_i - d_i(b_{-i}) = v_i + \sum_{j\neq i} b_j - d_i(b_{-i}) & \text{if } \sum_{i=1}^n b_i \geqq 0 \\ -d_i(b_{-i}) & \text{otherwise} \end{cases} \tag{9.8}$$

For exactly the same reason as for the mechanism above, one can prove that it is optimal for each agent i to report his true net value.

If the function $d_i(b_{-i})$ is suitably chosen, the size of the side-payment can be significantly reduced. One nice choice is the so-called *Clark mechanism* (also called *pivotal mechanism*):

The *Pivotal Mechanism* is a special case of the general Groves Mechanism in which $d_i(b_{-i})$ is given by

$$d_i(b_{-i}) = \begin{cases} \sum_{j\neq i} b_j & \text{if } \sum_{j\neq i} b_j \geqq 0 \\ 0 & \text{if } \sum_{j\neq i} b_j < 0 \end{cases}$$

In this case, it gives

$$t_i(b) = \begin{cases} 0 & \text{if } \sum_{i=1}^n b_i \geqq 0 \text{ and } \sum_{j\neq i} b_i \geqq 0, \\ \sum_{j\neq i} b_j & \text{if } \sum_{i=1}^n b_i \geqq 0 \text{ and } \sum_{j\neq i} b_j < 0 \\ -\sum_{j\neq i} b_j & \text{if } \sum_{i=1}^n b_i < 0 \text{ and } \sum_{j\neq i} b_j \geqq 0 \\ 0 & \text{if } \sum_{i=1}^n b_i < 0 \text{ and } \sum_{j\neq i} b_j < 0 \end{cases} \tag{9.9}$$

i.e.,

$$t_i(b) = \begin{cases} -|\sum_{j\neq i} b_i| & \text{if } (\sum_{i=1}^n b_i)(\sum_{j\neq i} b_i) < 0 \\ -|\sum_{j\neq i} b_i| & \text{if } \sum_{i=1}^n b_i = 0 \text{ and } \sum_{j\neq i} b_i < 0 \\ 0 & \text{otherwise} \end{cases} \tag{9.10}$$

Therefore, the payoff of agent i

$$\phi_i(b) = \begin{cases} v_i & \text{if} \sum_{i=1}^n b_i \geqq 0 \quad \text{and} \sum_{j\neq i} b_j \geqq 0 \\ v_i + \sum_{j\neq i} b_j & \text{if} \sum_{i=1}^n b_i \geqq 0 \quad \text{and} \sum_{j\neq i} b_j < 0 \\ -\sum_{j\neq i} b_j & \text{if} \sum_{i=1}^n b_i < 0 \quad \text{and} \sum_{j\neq i} b_j \geqq 0 \\ 0 & \text{if } \sum_{i=1}^n b_i < 0 \quad \text{and} \sum_{j\neq i} b_j < 0 \end{cases} \tag{9.11}$$

Remark 9.7.1 Thus, from the transfer given in (9.10), adding in the side-payment has the effect of taxing agent i only if he changes the social decision. Such an agent is called the pivotal person. The amount of the tax agent i must pay is the amount by which agent i's bid damages the other agents. The price that agent i must pay to change the amount of public good is equal to the harm that he imposes on the other agents.

9.7.2 The Groves-Clark-Vickery Mechanism with Continuous Public Goods

Now we are concerned with the provision of continuous public goods. Consider a public goods economy with n agents, one private good, and K public goods. Denote

x_i: the consumption of the private good by i;

y: the consumption of the public goods by all individuals;

t_i: transfer payment to i;

$g_i(y)$: the contribution made by i;

$c(y)$: the cost function of producing public goods y that satisfies the condition:

$$\sum g_i(y) = c(y).$$

Then, agent i's budget constraint should satisfy

$$x_i + g_i(y) = w_i + t_i \tag{9.12}$$

and his utility functions are given by

$$\bar{u}_i(x_i, y) = x_i - w_i + u_i(y) \tag{9.13}$$

By substitution,

$$\begin{aligned} u_i(t_i, y) &= t_i - (u_i(y) - g_i(y)) \\ &\equiv t_i + v_i(y) \end{aligned}$$

where $v_i(y)$ is called the valuation function of agent i. From the budget constraint,

$$\sum_{i=1}^{n} \{x_i + g_i(y)\} = \sum_{1=i}^{n} w_i + \sum_{1=i}^{n} t_i \tag{9.14}$$

we have

$$\sum_{i=1}^{n} x_i + c(y) = \sum_{i=1}^{n} w_i + \sum_{1=i}^{n} t_i \tag{9.15}$$

The feasibility (or balanced) condition then becomes

$$\sum_{i=1}^{n} t_i = 0 \tag{9.16}$$

Recall that Pareto efficient allocations are completely characterized by

$$\max \sum a_i \bar{u}_i(x_i, y)$$

$$s.t. \qquad \sum_{1=i}^{n} x_i + c(y) = \sum_{1=i}^{n} w_i$$

For quasi-linear utility functions it is easily seen that the weights a_i must be the same for all agents since $a_i = \lambda$ for interior Pareto efficient allocations (no income effect), the Lagrangian multiplier, and y is thus uniquely determined for the special case of quasi-linear utility functions $u_i(t_i, y) = t_i + v_i(y)$. Then, the above characterization problem becomes

$$\max_{t_i, y} \left[\sum_{i=1}^{n} (t_i + v_i(y)) \right] \tag{9.17}$$

or equivalently

(1)

$$\max_{y} \sum_{i=1}^{n} v_i(y)$$

(2) (feasibility condition): $\sum_{i=1}^{n} t_i = 0$

Then, the Lindahl-Samuelson condition is given by:

$$\sum_{i=1}^{n} v_i'(y) = 0.$$

that is,

$$\sum_{i=1}^{n} u_i'(y) = c'(y)$$

Thus, Pareto efficient allocations for quasi-linear utility functions are completely characterized by the Lindahl-Samuelson condition $\sum_{i=1}^{n} v_i'(y) = 0$ and feasibility condition $\sum_{i=1}^{n} t_i = 0$.

In the Groves mechanism, it is supposed that each agent is asked to report the valuation function $v_i(y)$. Denote his reported valuation function by $b_i(y)$.

To get the efficient level of public goods, the government may announce that it will provide a level of public goods y^* that maximizes

$$\max_y \sum_{i=1}^n b_i(y)$$

The Groves mechanism has the form:

$$\Gamma = (V, h) \tag{9.18}$$

where $V = V_1 \times \ldots \times V_n$: is the message space that consists of the set of all possible valuation functions with element $b_i(y) \in V_i$, $h = (t_1(b), t_2(b), \ldots, t_n(b), y(b))$ are outcome functions. It is determined by:

(1) Ask each agent i to report his/her valuation function $b_i(y)$ which may or may not be the true valuation $v_i(y)$.

(2) Determine y^*: the level of the public goods, $y^* = y(b)$, is determined by

$$\max_y \sum_{i=1}^n b_i(y) \tag{9.19}$$

(3) Determine t: transfer of agent i, t_i is determined by

$$t_i = \sum_{j \neq i} b_j(y^*) \tag{9.20}$$

The payoff of agent i is then given by

$$\phi_i(b(y^*)) = v_i(y^*) + t_i(b) = v_i(y^*) + \sum_{j \neq i} b_j(y^*) \tag{9.21}$$

The social planner's goal is to have the optimal level y^* that solves the problem:

$$\max_y \sum_{i=1}^n b_i(y).$$

In which case is the individual's interest consistent with social planner's interest? Under the rule of this mechanism, it is optimal for each agent i to truthfully report his true valuation function $b_i(y) = v_i(y)$ since agent i wants to maximize

$$v_i(y) + \sum_{j \neq i} b_j(y).$$

By reporting $b_i(y) = v_i(y)$, agent i ensures that the government will choose y^* which also maximizes his payoff while the government maximizes the social welfare. That is, individual's interest is consistent with the social interest that is determined by the Lindahl-Samuelson condition. Thus, truth-telling, $b_i(y) = v_i(y)$), is a dominant strategy equilibrium.

In general, $\sum_{i=1}^n t_i(b(y)) \neq 0$, which means that a Groves mechanism in general does not result in Pareto efficient outcomes even if it satisfies the Lindahl-Samuelson condition, i.e., it is Pareto efficient to provide the public goods.

As in the discrete case, the total transfer can be very large, just as before, they can be reduced by an appropriate side-payment. The Groves mechanism can be modified to

$$t_i(b) = \sum_{j \neq i} b_j(y) - d(b_{-i}).$$

The general form of the Groves Mechanism is then $\Gamma =< V, t, y(b) >$ such that

(1) $\sum_{i=1}^n b_i(y(b)) \geqq \sum_{i=1}^n b_i(y)$ for $y \in Y$;

(2) $t_i(b) = \sum_{j \neq i} b_j(y) - d(b_{-i})$.

A special case of the Groves mechanism is independently described by Clark and is called the Clark mechanism (also called the pivotal mechanism) in which $d_i(b_{-i}(y))$ is given by

$$d(b_{-i}) = \max_y \sum_{j \neq i} b_j(y). \tag{9.22}$$

That is, the pivotal mechanism, $\Gamma =< V, t, y(b) >$, is to choose (y^*, t_i^*) such that

(1) $\sum_{i=1}^n b_i(y^*) \geqq \sum_{i=1}^n b_i(y)$ for $y \in Y$;

(2) $t_i(b) = \sum_{j \neq i} b_j(y^*) - \max_y \sum_{j \neq i} b_j(y)$.

It is interesting to point out that the Clark mechanism contains the well-known Vickery auction mechanism (the second-price auction mechanism) as a special case. Under the Vickery mechanism, the highest biding person obtains the object, and he pays the second highest biding price. To see this, let us explore this relationship in the case of a single good auction (Example 9.3.3 in the beginning of this chapter). In this case, the outcome space is

$$Z = \{y \in \{0, 1\}^n : \sum_{i=1}^n y_i = 1\}$$

where $y_i = 1$ implies that agent i gets the object, and $y_i = 0$ means the person does not get the object. Agent i's valuation function is then given by

$$v_i(y) = v_i y_i.$$

Since we can regard y as a n-dimensional vector of public goods, by the Clark mechanism above, we know that

$$y = g(b) = \{y \in Z : \max \sum_{i=1}^{n} b_i y_i\} = \{y \in Z : \max_{i \in N} b_i\},$$

and the truth-telling is a dominate strategy. Thus, if $g_i(v) = 1$, then $t_i(v) = \sum_{j \neq i} v_j y_j^* - \max_y \sum_{j \neq i} v_j y_j = -\max_{j \neq i} v_j$. If $g_i(b) = 0$, then $t_i(v) = 0$. This means that the object is allocated to the individual with the highest valuation and he pays an amount equal to the second highest valuation. No other payments are made. This is exactly the outcome predicted by the Vickery mechanism.

9.8 Nash Implementation

9.8.1 Nash Equilibrium and General Mechanism Design

From Hurwicz's impossibility theorem, we know that, if one wants to get a mechanism that results in Pareto efficient and individually rational allocations, one must give up the dominant strategy implementation, and then, by Revelation Principle, we must look at more general mechanisms $< M, h >$ instead of using a revelation mechanism.

We know the dominant strategy equilibrium is a very strong solution concept. Now, if we adopt the Nash equilibrium as a solution concept to describe individuals' self-interested behavior, can we design an incentive compatible mechanism which implements Pareto efficient allocations?

For $e \in E$, a mechanism $< M, h >$ is said to have a *Nash equilibrium* $m^* \in M$ if

$$h_i(m^*) \succcurlyeq_i h_i(m_i, m_i^*) \qquad (9.23)$$

for all $m_i \in M_i$ and all i. Denote by $NE(e, \Gamma)$ the set of all Nash equilibria of the mechanism Γ for $e \in E$.

It is clear every dominant strategy equilibrium is a Nash equilibrium, but the converse may not be true.

A mechanism $\Gamma =< M, h >$ is said to Nash-implement a social choice correspondence F on E if for every $e \in E$

(a) $NE(e, \Gamma) \neq \emptyset$;

(b) $h(NE(e, \Gamma)) \subseteq F(e)$.

It fully Nash implements a social choice correspondence F on E if for every $e \in E$

(a) $NE(e, \Gamma) \neq \emptyset$

(b) $h(NE(e, \Gamma)) = F(e)$.

The following proposition shows that, if a truth-telling about their characteristics is a Nash equilibrium of the revelation mechanism, it must be a dominant strategy equilibrium of the mechanism.

Proposition 9.8.1 *For a revelation mechanism $\Gamma =< E, h >$, a truth-telling e^* is a Nash equilibrium if and only if it is a dominant equilibrium*

$$h(e_i^*, e_{-i}) \succcurlyeq_i h(e_i, e_{-i}) \quad \forall (e_i, e_{-i}) \in E \& i \in N. \tag{9.24}$$

Proof. Since for every $e \in E$ and i, by Nash equilibrium, we have

$$h(e_i, e_{-i}) \succcurlyeq_i h(e_i', e_{-i}) \quad \text{for all } e_i' \in E_i.$$

Since this true for any (e_i', e_{-i}), it is a dominant strategy equilibrium. The proof is completed.

Thus, we cannot get any new results if one insists on the choice of revelation mechanisms. To get more satisfactory results, one must give up the revelation mechanism, and look for a more general mechanism with general message spaces.

Notice that, when one adopts the Nash equilibrium as a solution concept, the weak Nash implementation is not a useful concept. To see this, consider any social choice correspondence F and the following mechanism: each individual's message space consists of the set of economic environments, i.e., it is given by $M_i = E$. The outcome function is defined as $h(m) = a \in F(e)$ when all agents reports the same economic environment $m_i = e$, and otherwise it is seriously punished by giving a worse outcome. Then, it is clear the truth-telling is a Nash equilibrium. However, it has a lot of Nash equilibria, in fact infinity number of Nash equilibria. Any false reporting about the economic environment $m_i = e'$ is also a Nash equilibrium. So, when we use Nash equilibrium as a solution concept, we need a social choice rule to be implemented or full implemented in Nash equilibrium.

9.8.2 Characterization of Nash Implementation

Now we discuss what kind of social choice rules can be implemented through Nash incentive compatible mechanism. Maskin in 1977 gave necessary and sufficient conditions for a social choice rule to be Nash implementable (This paper was not published till 1999. It then appeared in *Review of Economic Studies, 1999*). Maskin's result is fundamental since it not only helps us to understand what kind of social choice correspondence can be Nash implemented, but also gives basic techniques and methods for studying if a social choice rule is implementable under other solution concepts.

Maskin's monotonicity condition can be stated in two different ways although they are equivalent.

Definition 9.8.1 (Maskin's Monotonicity) A social choice correspondence $F : E \rightarrow\rightarrow A$ is said to be Maskin's monotonic if for any $e, \bar{e} \in E$, $x \in F(e)$ such that for all i and all $y \in A$, $x \succcurlyeq_i y$ implies that $x \bar{\succcurlyeq}_i y$, then $x \in F(\bar{e})$.

In words, Maskin's monotonicity requires that if an outcome x is socially optimal with respect to economy e and then economy is changed to $\bar{e}$ so that in each individual's ordering, x does not fall below an outcome that is not below before, then x remains socially optimal with respect to $\bar{e}$.

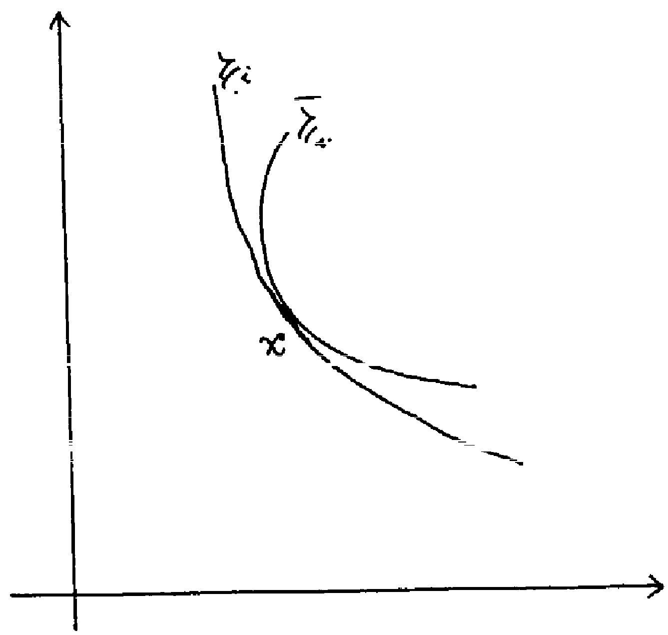

Figure 9.3: An illustration of Maskin's monotonicity.

Definition 9.8.2 (Another Version of Maskin's Monotonicity) A equivalent condition for a social choice correspondence $F : E \to\to A$ to be Maskin's monotonic is that, if for any two economic environments $e, \bar{e} \in E$, $x \in F(e)$ such that $x \notin F(e')$, there is an agent i and another $y \in A$ such that $x \succcurlyeq_i y$ and $y \bar{\succ}_i x$.

Maskin's monotonicity is a reasonable condition, and many well known social choice rules satisfy this condition.

Example 9.8.1 (Weak Pareto Efficiency) The weak Pareto optimal correspondence $P_w : E \to\to A$ is Maskin's monotonic.

Proof. If $x \in P_w(e)$, then for all $y \in A$, there exists $i \in N$ such that $x \succcurlyeq_i y$. Now if for any $j \in N$ such that $x \succcurlyeq_j$ implies $x \bar{\succcurlyeq}_j y$, then we have $x \bar{\succcurlyeq}_i y$ for particular i. Thus, $x \in P_w(\bar{e})$.

Example 9.8.2 (Majority Rule) The majority rule or call the Condorcet correspondence $CON : E \to\to A$ for strict preference profile, which is defined by

$$CON(e) = \{x \in A : \#\{i|x \succ_i y\} \geqq \#\{i|y \succ_i x\} \text{ for all } y \in A\}$$

is Maskin's monotonic.

Proof. If $x \in CON(e)$, then for all $y \in A$,

$$\#\{i|x \succ_i y\} \geqq \#\{i|y \succ_i x\}. \tag{9.25}$$

But if $\bar{e}$ is an economy such that, for all i, $x \succ_i y$ implies $x \bar{\succ}_i y$, then the left-hand side of (9.25) cannot fall when we replace e by $\bar{e}$. Furthermore, if the right-hand side rises, then we must have $x \succ_i y$ and $y \bar{\succ}_i x$ for some i, a contradiction of the relation between e and $\bar{e}$, given the strictness of preferences. So x is still a majority winner with respect to $\bar{e}$, i.e., $x \in CON(\bar{e})$.

In addition to the above two examples, Walrasian correspondence and Lindahl correspondence with interior allocations are Maskin's monotonic. The class of preferences that satisfy "single-crossing" property and individuals' preferences over lotteries satisfy the von Neumann-Morgenstern axioms also automatically satisfy Maskin' monotonicity.

The following theorem shows the Maskin's monotonicity is a necessary condition for Nash-implementability.

Theorem 9.8.1 *For a social choice correspondence $F : E \to\to A$, if it is fully Nash implementable, then it must be Maskin's monotonic.*

Proof. For any two economies $e, \bar{e} \in E$, $x \in F(e)$, then by full Nash implementability of F, there is $m \in M$ such that m is a Nash equilibrium and $x = h(m)$. This means that $h(m) \succcurlyeq_i h(m_i', m_{-i})$ for all i and $m_i' \in M_i$. Given $x \succcurlyeq_i y$ implies $x \bar{\succcurlyeq}_i y$, $h(m) \bar{\succcurlyeq}_i h(m_i', m_{-i})$, which means that m is also a Nash equilibrium at $\bar{e}$. Thus, by Nash implementability again, we have $x \in F(\bar{e})$.

Maskin's monotonicity itself can not guarantee a social choice correspondence is fully Nash implementable. However, under the so-called no-veto power, it becomes sufficient.

Definition 9.8.3 (No-Veto Power) A social choice correspondence $F : E \to\to A$ is said to satisfy no-veto power if whenever for any i and e such that $x \succcurlyeq_j y$ for all $y \in A$ and all $j \neq i$, then $x \in F(e)$.

The no-veto power (NVP) condition implies that if $n - 1$ agents regard an outcome is the best to them, then it is social optimal. This is a rather weaker condition. NVP is satisfied by virtually all "standard" social choice rules, including weak Pareto efficient and

Condorect correspondences. It is also often automatically satisfied by any social choice rules when the references are restricted. For example, for private goods economies with at least three agents, if each agent's utility function is strict monotonic, then there is no other allocation such that $n-1$ agents regard it best, so the no-veto power condition holds.

Theorem 9.8.2 *Under no-veto power, if Maskin's monotonicity condition is satisfied, then F is fully Nash implementable.*

Proof. The proof is by construction. For each agent i, his message space is defined by

$$M_i = E \times A \times \mathcal{N}$$

where $\mathcal{N} = \{1, 2, \ldots, \}$. Its elements are denoted by $m_i = (e_i, a_i, v_i)$, which means each agent i announces an economic profile, an outcome, and a real number. (For notation convenience, we have used e_i to denote the economic profile of all individuals' economic characteristics, but not just agent i economic characteristic).

The outcome function is constructed in three cases:

Case(1). If $m_1 = m_2 = \ldots = m_n = (e, a, v)$ and $a \in F(e)$, the outcome function is defined by

$$h(m) = a.$$

In words, if players are unanimous in their strategy, and their proposed alterative a is F-optimal, given their proposed profile e, the outcome is a.

Case(2). For all $j \neq i$, $m_j = (e, a, v)$, $m_i = (e_i, a_i, v_i) \neq (e, a, v)$, and $a \in F(e)$, define:

$$h(m) = \begin{cases} a_i & \text{if } a_i \in L(a, e_i) \\ a & \text{if } a_i \notin L(a, e_i). \end{cases}$$

where $L(a, e_i) = \{b \in A : a \ R_i \ b\}$ which is the lower contour set of R_i at a. That is, suppose that all players but one play the same strategy and, given their proposed profile, their proposed alternative a is F-optimal. Then, the odd-man-out, gets his proposed alterative, provided that it is in the lower contour set at a of the ordering that the other players propose for him; otherwise, the outcome is a.

Case(3). If neither Case (1) nor Case (2) applies, then define

$$h(m) = a_{i*}$$

where $i* = \max\{i \in \mathcal{N} : v_i = \max_j v_j\}$. In other words, when neither case (1) nor case (2) applies, the outcome is the alternative proposed by player with the highest index among those whose proposed number is maximal.

Now we show that the mechanism $\langle M, h \rangle$ defined above fully Nash implements social choice correspondence F, i.e., $h(N(e)) = F(e)$ for all $e \in E$. We first show that $F(e) \subset h(N(e))$ for all $e \in E$, i.e., we need to show that for all $e \in E$ and $a \in F(e)$, there exists a $m \in M$ such that $a = h(m)$ is a Nash equilibrium outcome. To do so, we only need to show that any m which is given by Case (1) is a Nash equilibrium. Note that $h(m) = a$ and for any given $m_i' = (e_i', a_i', v_i') \neq m_i$, by Case (2), we have

$$h(m_i', m_{-i}) = \begin{cases} a_i & \text{if } a_i \in L(a, e_i) \\ a & \text{if } a_i \notin L(a, e_i). \end{cases}$$

and thus

$$h(m) \; R_i \; h(m_i', m_{-i}) \quad \forall m_i' \in M_i.$$

Hence, m is a Nash equilibrium.

We now show that for each economic environment $e \in E$, if m is a Nash equilibrium, then $h(m) \in F(e)$. First, consider the Nash equilibrium m is given by Case (1) so that $a \in F(e)$, but the true economic profile is e', i.e., $m \in NE(e', \Gamma)$. We need to show $a \in F(e')$. By Case (1), $h(m) = a$. Since a is a Nash equilibrium outcome with respect to e', by Case (2), for all $i \in N$ and $b \in L(a, e)$, we have $a \; R_i' \; b$, which can be rewritten as: for $i \in N$ and $b \in A$, $a \; R_i \; b$ implies $a \; R_i' \; b$. Thus, by Maskin's monotonicity condition, we have $a \in F(e')$.

Next, suppose Nash equilibrium m for e' is in the Case (2), i.e, for all $j \neq i$, $m_j = (e, a, v)$, $m_i \neq (e, a, v)$. Let $a' = h(m)$. By Case (3), each $j \neq i$ can induce any outcome $b \in A$ by choosing (R_j, b, v_j) with sufficiently a large v_j(which is greater than $\max_{k \neq j} v_k$), as the outcome at (m_i', m_{-i}), i.e,, $b = h(m_i', m_{-i})$. Hence, m is a Nash equilibrium with respect to e' implies that for all $j \neq i$, we have

$$a' \; R_j' \; b.$$

Thus, by no-veto power assumption, we have $a' \in F(e')$.

The same argument as the above, if m is a Nash equilibrium for e' is given by Case (3), we have $a' \in F(e')$. The proof is completed.

Although Maskin's monotonicity is very weak, it is violated by some social choice rules such as Solomon's famous judgement. Solomon's solution falls under Nash equilibrium implementation, since each woman knows who is the real mother. His solution, which consisted in threatening to cut the baby in two, is not entirely foolproof. What would be have done if the impostor has had the presence of mind to scream like a real mother? Solomon's problem can be formerly described by the languages of mechanism design as follows.

Two women: Anne and Bets

Two economies (states): $E = \{\alpha, \beta\}$, where

α: Anne is the real mother

β: Bets is the real mother

Solomon has three alternatives so that the feasible set is given by $A = \{a, b, c\}$, where

a: give the baby to Anne

b: give the baby to Bets

c: cut the baby in half.

Solomon's desirability (social goal) is to give the baby to the real mother,

$f(\alpha) = a$ if α happens

$f(\beta) = b$ if β happens

Preferences of Anne and Bets:

For Anne,

at state α, $a \succ_A^\alpha b \succ_A^\alpha c$

at β: $a \succ_A^\beta c \succ_A^\beta b$

For Bets,

at state α, $b \succ_B^\alpha c \succ_B^\alpha a$

at state β: $b \succ_B^\beta a \succ_B^\beta c$

To see Solomon's solution does not work, we only need to show his social choice goal is not Maskin's monotonic. Notice that for Anne, since

$$a \succ_A^{\alpha} b, c,$$

$$a \succ_A^{\beta} b, c,$$

and $f(\alpha) = a$, by Maskin's monotonicity, we should have $f(\beta) = a$, but we actually have $f(\beta) = b$. So Solomon's social choice goal is not Nash implementable.

9.9 Better Mechanism Design

Maskin's theorems gives necessary and sufficient conditions for a social choice correspondence to be Nash implementable. However, due to the general nature of the social choice rules under consideration, the implementing mechanisms in proving characterization theorems turn out to be quite complex. Characterization results show what is possible for the implementation of a social choice rule, but not what is realistic. Thus, like most characterization results in the literature, Maskin's mechanism is not natural in the sense that it is not continuous; small variations in an agent's strategy choice may lead to large jumps in the resulting allocations, and further it has a message space of infinite dimension since it includes preference profiles as a component. In this section, we give some mechanisms that have some desired properties.

9.9.1 Groves-Ledyard Mechanism

Groves-Ledyard Mechanism (1977, Econometrica) was the first to give a specific mechanism that Nash implements Pareto efficient allocations for public goods economies.

To show the basic structure of the Groves-Ledyard mechanism, consider a simplified Groves-Ledyard mechanism. Public goods economies under consideration have one private good x_i, one public good y, and three agents ($n = 3$). The production function is given by $y = v$.

The mechanism is defined as follows:

$M_i = R_i$, $i = 1, 2, 3$. Its elements, m_i, can be interpreted as the proposed contribution (or tax) that agent i is willing to make.

$t_i(m) = m_i^2 + 2m_j m_k$: the actual contribution t_i determined by the mechanism with the reported m_i.

$y(m) = (m_1 + m_2 + m_3)^2$: the level of public good y.

$x_i(m) = w_i - t_i(m)$: the consumption of the private good.

Then the mechanism is balanced since

$$\begin{aligned} & \sum_{i=1}^{3} x_i + \sum_{i=1}^{3} t_i(m) \\ = & \sum_{i=1}^{3} x_i + (m_1 + m_2 + m_3)^2 \\ = & \sum_{i=3}^{n} x_i + y = \sum_{i=1}^{3} w_i \end{aligned}$$

The payoff function is given by

$$\begin{aligned} v_i(m) &= u_i(x_i(m), y(m)) \\ &= u_i(w_i - t_i(m), y(m). \end{aligned}$$

To find a Nash equilibrium, we set

$$\frac{\partial v_i(m)}{\partial m_i} = 0 \tag{9.26}$$

Then,

$$\frac{\partial v_i(m)}{\partial m_i} = \frac{\partial u_i}{\partial x_i}(-2m_i) + \frac{\partial u_i}{\partial y} 2(m_1 + m_2 + m_3) = 0 \tag{9.27}$$

and thus

$$\frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = \frac{m_i}{m_1 + m_2 + m_3}. \tag{9.28}$$

When u_i are quasi-concave, the first order condition will be a sufficient condition for Nash equilibrium.

Making summation, we have at Nash equilibrium

$$\sum_{i=1}^{3} \frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = \sum \frac{m_i}{m_1 + m_2 + m_3} = 1 = \frac{1}{f'(v)} \tag{9.29}$$

that is,

$$\sum_{i=1}^{3} MRS_{yx_i} = MRTS_{yv}.$$

Thus, the Lindahl-Samuelson and balanced conditions hold which means every Nash equilibrium allocation is Pareto efficient.

They claimed that they have solved the free-rider problem in the presence of public goods. However, economic issues are usually complicated. Some agreed that they indeed solved the problem, some did not. There are two weakness of Groves-Ledyard Mechanism: (1) it is not individually rational: the payoff at a Nash equilibrium may be lower than at the initial endowment, and (2) it is not individually feasible: $x_i(m) = w_i - t_i(m)$ may be negative.

How can we design the incentive mechanism to pursue Pareto efficient and individually rational allocations?

9.9.2 Walker's Mechanism

Walker (1981, Econometrica) gave such a mechanism. Again, consider public goods economies with n agents, one private good, and one public good, and the production function is given by $y = f(v) = v$.

The mechanism is defined by:

$M_i = R$

$Y(m) = \sum_{i=1}^{n} m_i$: the level of public good.

$q_i(m) = \frac{1}{n} + m_{i+1} - m_{i+2}$: personalized price of public good.

$t_i(m) = q_i(m)y(m)$: the contribution (tax) made by agent i.

$x_i(m) = w_i - t_i(m) = w_i - q_i(m)y(m)$: the private good consumption.

Then, the budget constraint holds:

$$x_i(m) + q_i(m)y(m) = w_i \forall m_i \in M_i \tag{9.30}$$

Making summation, we have

$$\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} q_i(m)y(m) = \sum_{i=1}^{n} w_i$$

and thus

$$\sum_{i=1}^{n} x_i + y(m) = \sum_{i=1}^{n} w_i$$

which means the mechanism is balanced.

The payoff function is

$$\begin{aligned} v_i(m) &= u_i(x_i, y) \\ &= u_i(w_i - q_i(m)y(m), y(m)) \end{aligned}$$

The first order conditions for interior allocations are given by

$$\begin{aligned} \frac{\partial v_i}{\partial m_i} &= -\frac{\partial u_i}{\partial x_i}\left[\frac{\partial q_i}{\partial m_i} y(m) + q_i(m)\frac{\partial y(m)}{\partial m_i}\right] + \frac{\partial u_i}{\partial y}\frac{\partial y(m)}{\partial m_i} \\ &= -\frac{\partial u_i}{\partial x_i} q_i(m) + \frac{\partial u_i}{\partial y} = 0 \\ &\Rightarrow \frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = q_i(m) \qquad \text{(FOC) for the Lindahl Allocation} \\ &\Rightarrow N(e) \subseteq L(e) \end{aligned}$$

Thus, if u_i are quasi-concave, it is also a sufficient condition for Lindahl equilibrium. We can also show every Lindahl allocation is a Nash equilibrium allocation, i.e.,

$$L(e) \subseteq N(e) \tag{9.31}$$

Indeed, suppose $[(x^*, y^*), q_i^*, \ldots, q_n^*]$ is a Lindahl equilibrium. The solution m^* of the following equation

$$\begin{aligned} q_i^* &= \frac{1}{n} + m_{i+1} - m_{i+2}, \\ y^* &= \sum_{i=1}^{n} m_i \end{aligned}$$

is a Nash equilibrium.

Thus, Walker's mechanism fully implements Lindahl allocations which are Pareto efficient and individually rational.

Walker's mechanism also has a disadvantage that it is not feasible although it does solve the individual rationality problem. If a person claims large amounts of t_i, consumption of private good may be negative, i.e., $x_i = w_i - t_i < 0$. Tian proposed a mechanism that overcomes Walker's mechanism's problem. Tian's mechanism is individually feasible, balanced, and continuous.

9.9.3 Tian's Mechanism

In Tian's mechanism (JET, 1991), everything is the same as Walker's, except that $y(m)$ is given by

$$y(m) = \begin{cases} a(m) & if \quad \sum m_i > a(m) \\ \sum(m_i) & if \quad 0 \leqq \sum m_i \leqq a(m) \\ 0 & if \quad \sum_{i=1}^{n} m_i < 0 \end{cases} \tag{9.32}$$

where $a(m) = \min_{i \in N'(m)} \frac{w_i}{q_i(m)}$ with $N'(m) = \{i \in N : q_i(m) > 0\}$.

An interpretation of this formulation is that if the total taxes that the agents are willing to pay were between zero and the feasible upper bound, the level of public good to be produced would be exactly the total taxes; if the total taxes were less than zero, no public good would be produced; if the total taxes exceeded the feasible upper bound, the level of the public good would be equal to the feasible upper bound.

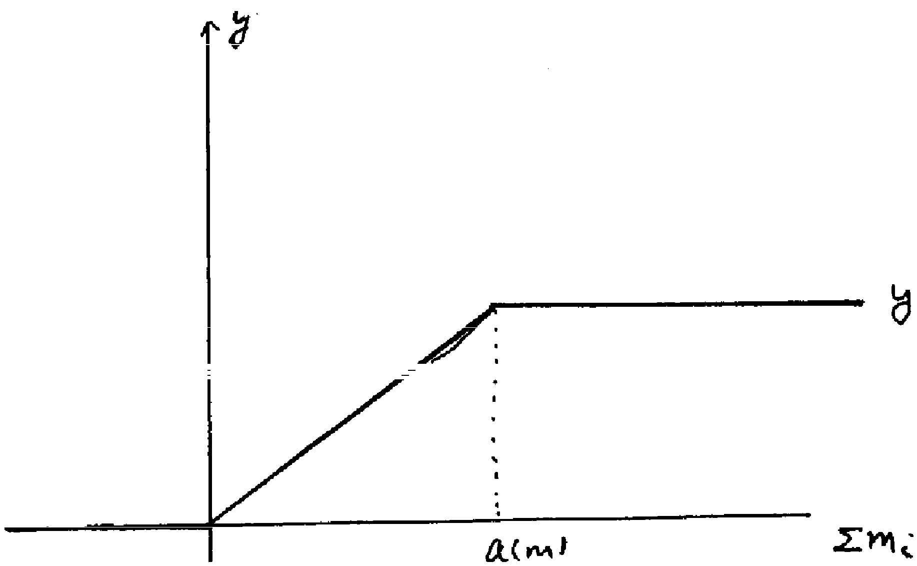

Figure 9.4: The feasible public good outcome function $Y(m)$.

To show this mechanism has all the nice properties, we need to assume that preferences are strictly monotonically increasing and convex, and further assume that every interior allocation is preferred to any boundary allocations: For all $i \in N, (x_i, y)\ P_i\ (x'_i, y')$ for all $(x_i, y) \in \mathbb{R}^2_{++}$ and $(x'_i, y') \in \partial\mathbb{R}^2_+$, where $\partial\mathbb{R}^2_+$ is the boundary of $\mathbb{R}^2_+$.

To show the equivalence between Nash allocations and Lindahl allocations. We first prove the following lemmas.

Lemma 9.9.1 *If $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, then $(X_i(m^*), Y(m^*)) \in \mathbb{R}^2_{++}$ for all $i \in N$.*

Proof: We argue by contradiction. Suppose $(X_i(m^*), Y(m^*)) \in \partial\mathbb{R}^2_+$. Then $X_i(m^*) = 0$ for some $i \in N$ or $Y(m^*) = 0$. Consider the quadratic equation

$$y = \frac{w^*}{2(y+c)}, \tag{9.33}$$

where $w^* = \min_{i\in N} w_i$; $c = b + n\sum_{i=1}^n |m_i^*|$, where $b = 1/n$. The larger root of (9.33) is positive and denoted by $\tilde{y}$. Suppose that player i chooses his/her message $m_i = \tilde{y} - \sum_{j\neq i}^n m_j^*$. Then $\tilde{y} = m_i + \sum_{j\neq i}^n m_j^* > 0$ and

$$\begin{aligned} w_j - q_j(m^*/m_i, i)\tilde{y} &\geqq w_j - [b + (n\sum_{s=1}^n |m_s^*| + \tilde{y})]\tilde{y} \\ &= w_j - (\tilde{y} + b + n\sum_{s=1}^n |m_s^*|)\tilde{y} \\ &= w_j - w^*/2 \geqq w_j/2 > 0 \end{aligned} \tag{9.34}$$

for all $j \in N$. Thus, $Y(m^*/m_i, i) = \tilde{y} > 0$ and $X_j(m^*/m_i, i) = w_j - q_j(m^*/m_i, i)Y(m^*/m_i, i) = w_j - q_j(m^*/m_i, i)\tilde{y} > 0$ for all $j \in N$. Thus we have $(X_i(m^*/m_i, i), Y(m^*/m_i, i)) \; P_i \; (X_i(m^*), Y(m^*))$ by Assumption 4, which contradicts the hypothesis $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

Lemma 9.9.2 *If $(X(m^*), Y(m^*)) \in N_{M,h}(e)$, then $Y(m^*)$ is an interior point of $[0, a(m)]$ and thus $Y(m^*) = \sum_{i=1}^n m_i^*$.*

Proof: By Lemma 9.9.1, $Y(m^*) > 0$. So we only need to show $Y(m^*) < a(m^*)$. Suppose, by way of contradiction, that $Y(m^*) = a(m^*)$. Then $X_j(m^*) = w_j - q_j(m^*)Y(m^*) = w_j - q_j(m^*)a(m^*) = w_j - w_j = 0$ for at least some $j \in N$. But $X(m^*) > 0$ by Lemma 9.9.1, a contradiction. Q.E.D.

Proposition 1 *If the mechanism has a Nash equilibrium m^*, then $(X(m^*), Y(m^*))$ is a Lindahl allocation with $(q_1(m^*), \ldots, q_n(m^*))$ as the Lindahl price vector, i.e., $N_{M,h}(e) \subseteq L(e)$.*

Proof: Let m^* be a Nash equilibrium. Now we prove that $(X(m^*), Y(m^*))$ is a Lindahl allocation with $(q_1(m^*), \ldots, q_n(m^*))$ as the Lindahl price vector. Since the mechanism is completely feasible and $\sum_{i=1}^n q_i(m^*) = 1$ as well as $X_i(m^*) + q_i(m^*)Y(m^*) = w_i$ for all $i \in N$, we only need to show that each individual is maximizing his/her preference. Suppose, by way of contradiction, that there is some $(x_i, y) \in \mathbb{R}^2_+$ such that

$(x_i, y)\ P_i\ (X_i(m^*), Y(m^*))$ and $x_i + q_i(m^*)y \leqq w_i$. Because of monotonicity of preferences, it will be enough to confine ourselves to the case of $x_i + q_i(m^*)y = w_i$. Let $(x_{i\lambda}, y_\lambda) = (\lambda x_i + (1\lambda)X_i(m^*), \lambda y + (1\lambda)Y(m^*))$. Then by convexity of preferences we have $(x_{i\lambda}, y_\lambda)\ P_i\ (X_i(m^*), Y(m^*))$ for any $0 < \lambda < 1$. Also $(x_{i\lambda}, y_\lambda) \in \mathbb{R}^2_+$ and $x_{i\lambda} + q_i(m^*)y_\lambda = w_i$.

Suppose that player i chooses his/her message $m_i = y_\lambda - \sum_{j\neq i}^n m_j^*$. Since $Y(m^*) = \sum_{j=1}^n m_i^*$ by Lemma 9.9.2, $m_i = y_\lambda - Y(m^*) + m_i^*$. Thus as $\lambda \to 0$, $y_\lambda \to Y(m^*)$, and therefore $m_i \to m_i^*$. Since $X_j(m^*) = w_j - q_j(m^*)Y(m^*) > 0$ for all $j \in N$ by Lemma 9.9.1, we have $w_j - q_j(m^*/m_i, i)y_\lambda > 0$ for all $j \in N$ as λ is a sufficiently small positive number. Therefore, $Y(m^*/m_i, i) = y_\lambda$ and $X_i(m^*/m_i, i) = w_i - q_i(m^*)Y(m^*/m_i, i) = w_i - q_i(m^*)y_\lambda = x_{i\lambda}$. From $(x_{i\lambda}, y_\lambda)\ P_i\ (X_i(m^*), Y(m^*))$, we have $(X_i(m^*/m_i, i), Y(m^*/m_i, i))\ P_i$ $(X_i(m^*), Y(m^*))$. This contradicts $(X(m^*), Y(m^*)) \in N_{M,h}(e)$. Q.E.D.

Proposition 2 *If (x^*, y^*) is a Lindahl allocation with the Lindahl price vector $q^* = (q_1^*, \ldots, q_n^*)$, then there is a Nash equilibrium m^* of the mechanism such that $X_i(m^*) = x_i^*$, $q_i(m^*) = q_i^*$, for all $i \in N$, $Y(m^*) = y^*$, i.e., $L(e) \subseteq N_{M,h}(e)$.*

Proof: We need to show that there is a message m^* such that (x^*, y^*) is a Nash allocation. Let m^* be the solution of the following linear equations system:??

$$
\begin{aligned}
q_i^* &= \frac{1}{n} + m_{i+1} - m_{i+2}, \\
y^* &= \sum_{i=1}^n m_i
\end{aligned}
$$

Then $X_i(m^*) = x_i^*$, $Y(m^*) = y^*$ and $q_i(m^*) = q_i^*$ for all $i \in N$. Thus from $(X(m^*/m_i, i), Y(m^*/m_i, i)) \in \mathbb{R}^2_+$ and $X_i(m^*/m_i, i) + q_i(m^*)Y(m^*/m_i, i) = w_i$ for all $i \in N$ and $m_i \in M_i$, we have $(X_i(m^*), Y(m^*))\ R_i\ (X_i(m^*/m_i, i), Y(m^*/m_i, i))$. Q.E.D.

Thus, Tian's mechanism Nash implements Lindahl allocations.

9.10 Incomplete Information and Bayesian Nash Implementation

Nash implementation has imposed a very strong assumption on information requirement. Although the designer does not know information about agents' characteristics, Nash equilibrium assume each agent knows characteristics of the others. This assumption is hardly satisfied in many cases in the real world. Can we remove this assumption? The answer is positive. One can use the Bayesian-Nash equilibrium, introduced by Harsanyi, as a solution concept to describe individuals' self-interested behavior. Although each agent does not know economic characteristics of the others, he knows the probability distribution of economic characteristics of the others. In this case, we can still design an incentive compatible mechanism.

For simplicity, assume the utility function of each agent is known up to one parameter, θ_i, so that it can be denoted by $u_i(x, \theta_i)$. Assume that all agents and the designer know that the vector of types, $\theta = (\theta_1, \ldots, \theta_n)$ is distributed according to $q(\theta)$ a priori on a set Θ.

Each agent knows his own type θ_i, and therefore computes the conditional distribution of the types of the other agents:

$$q(\theta_{-i}|\theta_i) = \frac{q(\theta_i, \theta_{-i})}{\int_{\Theta_{-i}q(\theta_i,\theta_{-i})d\theta_{-i}}}.$$

As usual, a mechanism is a pair, $\Gamma = \langle M, h\rangle$. Given a mechanism $\langle M, h\rangle$, the choice of m_i of each agent i is the function of θ_i: $\sigma_i : \Theta_i \to M_i$. Let Σ_i be the set of all such strategies of agent i. Given $\sigma = (\sigma_1, \ldots, \sigma_n)$, agent i's expected utility at t_i is given by

$$\Pi^i_{\langle M,h\rangle}(\sigma; \theta_i) = \int_{\Theta_{-i}} u_i(h(\sigma(\theta)), \theta_i) q(\theta_{-i}|\theta_i) d\theta_{-i}. \tag{9.35}$$

A strategy σ is a *Bayesian-Nash equilibrium* of $\langle M, h\rangle$ if for all $\theta_i \in \Theta_i$,

$$\Pi^i(\sigma; \theta_i) \geqq \Pi^i(\hat{\sigma}_i, \sigma_{-i}; \theta_i) \quad \forall \hat{\sigma}_i \in \Sigma_i.$$

Given a mechanism $\langle M, h\rangle$, the set of its Bayesian-Nash equilibria depends on economic environments, denoted by $B(e, \Gamma)$. Like Nash implementation, Bayesian-Nash incentive compatibility involves the relationship between $F(e)$and $B(e, \Gamma)$. If for all $e \in E$, $B(e)$ is a subset of $F(e)$, we say the mechanism $\langle M, h\rangle$ Bayesian-Nash implements social choice correspondence F. If for a given social choice correspondence F, there is some mechanism that Bayesian-Nash implements F, we call this social choice correspondence Bayesian-Nash implementable.

Pastlewaite–Schmeidler (JET, 1986), Palfrey-Srivastava (RES, 1989), Mookherjee-Reichelstein (RES, 1990), Jackson (Econometrica, 1991), Dutta-Sen (Econometrica, 1994), Tian (Social Choices and Welfare, 1996; Journal of Math. Econ., 1999) provided necessary and sufficient conditions for a social choice correspondence F to be Bayesian-Nash implementable. Under some technical conditions, they showed that a social choice correspondence F is Bayesian-Nash implementable if and only if F is Bayesian monotonic and Bayesian-Nash incentive compatible.

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