Valuation of Multi-Name Credit Derivatives

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Abstract

The subject of this thesis is modelling and valuation of multi-name credit derivatives such as basket default swaps and basket CDS tranches. After a description of the relevant products, models for valuation are presented. The key idea of modelling correlated default is the usage of copulae. In this thesis the valuation models are set up with Gaussian normal-, Student t- and Clayton copulae. Two different methods for valuation are described: The first is the standard Monte Carlo method for simulating the default times, with which multi-name credit derivatives can be priced. In the second approach the correlation structure is simplified by a factor copula model, in which semi-explicit fomulae for valuation of the multi-name credit products can be formulated.

In this thesis the Monte Carlo valuation approach is implemented for the Gaussian normal-, the Student t- and the Clayton copula; the semi-explicit approach is implemented for the Gaussian normal copula.

The implemented valuation methods are used to price a simple three credit basket and a real world basket CDS tranche. It is investigated what influence parameters like correlation and recovery rates have on the spread of credit products. For the Student t- and Clayton copula correlation structure the dependence of the spread on the copula parameters ν and α is examined.

Contents

Chapter 1

Introduction

In recent years the credit derivatives market has grown rapidly in volume and complexity of the products offered. While the biggest part of outstanding notionals is still found in simple products like credit default swaps, complex products having payoff profiles depending on a whole credit portfolio are becoming more popular. Examples of such products are basket default swaps (BDS) and basket CDS tranches. A basket default swap extends the credit protection, which a simple credit default swap (CDS) grants for a single underlying, to a portfolio of underlyings with the restriction that the default of only one underlying is compensated. Depending on the ranking of the protected default, the product is called 1st-to-default basket, 2nd-to-default basket or, generally, k^{th}-to-default basket. A basket CDS tranche is a generalization of the basket default swap in the sense that a certain percentile of the portfolio notional is protected. The price of such products depends on the joint default probability of the underlyings in the credit portfolio. Modelling joint default is difficult, as observations of joint defaults are even scarcer than the already rare event of a single default. Thus, modelling the joint distribution is the crucial point in pricing multi-name credit derivatives.

The thesis is structured as follows: The subsequent sections of this chapter contain an outline of the relevant credit models as well as of the products of interest; these are single-name and multi-name credit derivatives. Chapter 2 contains a description of the methods used for valuation of multi-name credit derivatives. The implementation of the valuation methods is given in Chapter 3. The implemented programs are used to investigate the influence of different parameters on a simple basket default swap with an underlying portfolio of three credits and a real world basket CDS tranche. The results are presented in Chapter 4.

1.1 Credit Risk Models

Starting in 1974, modelling credit risk has been of central interest to theorists as well as to practitioners, and different credit risk models have been developed. In the following sections a short overview over the fundamental ideas of the different models is given.

1.1.1 Merton's Structural Approach to Credit Risk

In the structural approach, which was developed by Merton in 1974 [16] and which is based on the principles of Black and Scholes' option pricing theory [4], the asset value V of a company is modelled as Brownian motion where the change in the asset value dV is given by:

$$dV = \mu V \, dt + V \, dW \tag{1.1}$$

The parameters μ and σ denote the drift and the volatility of the Brownian motion. The liabilities of a company are summarized in a zero coupon bond with face value F and maturity T. If the value of the assets V at maturity T is smaller than the value of the liabilities F, the company cannot meet its obligation to pay the liabilities and the company defaults. If the value of the assets V is greater than the liabilities F, then the company can fulfil its obligation to pay its liabilities at maturity T and no default occurs;

$$
\begin{array}{ll}
V > F : & \text{No Default} \\
V \leq F : & \text{Default}
\end{array}
$$

In this model the firm's equity can be regarded as an European call option on its assets with a strike of F and maturity T. In Merton's model default can occur only at maturity T of the zero coupon bond. An extension of Merton's model - that default can occur before maturity if the asset value V falls below a certain threshold D - was provided by Black and Cox [3]. A commercial product which follows this approach is CreditGrades [20]: The value of the assets $V(t)$ is modelled as random walk (1.1) with zero drift and default occurs if the value of the assets hits a certain lower threshold RD for the first time (with recovery R and debt per share D). RD follows a lognormal random walk with mean $\bar{R}D$ and standard deviation $Stdev(\ln(RD)) = \lambda$.

$$
\begin{array}{ll}
V(t) > RD(t) : & \text{No Default at time } t \\
V(t) \leq RD(t) : & \text{Default at time } t
\end{array}
$$

The default time τ is given by the first time the condition $V(t) \leq RD(t)$ is satisfied. Summarizing the description, the firm is modelled as a "down-and-out" barrier option with a time dependent knock-out barrier RD. Using the known distributions for the first stopping time of a Brownian motion, the survival probability $S(t) = 1 - F(t) = \mathbb{P}(\tau > t)$ is

$$S(t) = \Phi\left(\frac{\ln(d)}{A_t} - \frac{A_t}{2}\right) - d\,\Phi\left(-\frac{\ln(d)}{A_t} - \frac{A_t}{2}\right) \qquad (1.2)$$

with Φ denoting the cumulative standard normal distribution function, the parameters $A_t^2 = \sigma t + \lambda^2$ and $d = \frac{V(0)}{RD}e^{\lambda^2}$ (σ denotes the asset volatility and λ the barrier volatility).

Latent Variable Models Models where the default event depends on the evolution of a fundamental property of the considered company, are called latent variable models. The models developed by KMV [11] and CreditMetrics [17] are such latent variable models. In these models the relative change in the latent variables in a portfolio consisting of m debtors is given by a m-dimensional random vector $X = (X_1, ..., X_m)^T$ where the random variables X_i follow the related marginal distributions and the correlations between the latent variables. For each debtor i a lower threshold D_i (of the relative change of the latent variables) is given and when hit this leads to default of the debtor. As an indicator whether a debtor has defaulted or not, a default indicator Y_i is used which has a value of 1 in case of default and 0 in case of no default.

$$Y_i = 1 \iff X_i \leq D_i \qquad (1.3)$$

The cumulative distribution $\mathbb{P}(X_i \leq D_i)$ gives the probability that the random variable X_i is less or equal to the threshold D_i. Given historic time dependent default probabilities[1] the threshold D_i can be used to calibrate the cumulative distribution to the given historic default probabilities; for example, if X_i follows a normal distribution with a time dependent default probability of $F_i(t)$, the threshold equals $\Phi^{-1}(F_i(t))$. That is, default occurs if $X_i \leq \Phi^{-1}(F_i(t))$,[2] where Φ^{-1} is the inverse normal cumulative distribution.

---

[1] The default probability $F(t)$ is defined as the probability that the default time τ is smaller than or equal to time t: $F(t) = \mathbb{P}(\tau \leq t)$. The condition $X_i \leq D_i$ at time t is equivalent to the condition $\tau_i \leq t$.

[2] As X_i is normally distributed, the probability for condition $X_i \leq D_i$ is $\mathbb{P}(X_i \leq D_i) = \Phi(D_i)$. This probability is equal to $\mathbb{P}(\tau_i \leq t) = F_i(t)$ so that $\Phi(D_i) = F_i(t)$ which provides the sought threshold $D_i = \Phi^{-1}(F_i(t))$.

This last step of linking the thresholds to default probabilities is not part of standard Merton's model. Some authors, like Mashal and Naldi [14], name this approach "hybrid approach". The name derives from the fact that this approach uses typical features of Merton's approach (default happens if V falls below a certain threshold D) as well as typical features of the hazard rate model (calibration to market implied default probabilities).

If the evolution of the asset values of several companies is considered, the asset values of the companies cannot change independently as each company acts in the same macroeconomic environment which has influence on the evolution of the asset values. Thus the Wiener processes W, which determine the random walk of the asset value V, cannot be independent but have to be correlated. As the value of the latent variable relative to the threshold determines whether a firm defaults or not, the dependence structure of default is given by the dependence structure of the latent variables.

Correlation structure in Merton's Approach In Merton's approach it is not only assumed that the marginal distribution of the latent variables is a normal distribution but also that the probability of several debtors defaulting can be modelled by a multi-variate Gaussian distribution. For a two debtor universe the probability that debtor 1 and debtor 2 default is given by the two dimensional Gaussian distribution:

$$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) = \int\limits_{-\infty}^{x_1} \int\limits_{-\infty}^{x_2} \frac{1}{2\pi\sqrt{1-\rho_{12}^2}} \exp\left(-\frac{s^2 - 2\rho_{12}st + t^2}{2(1-\rho_{12}^2)}\right) ds\,dt \quad (1.4)$$

In the equation above ρ_{12} is the correlation between default of credit 1 and default of credit 2.

In order to determine in Merton's model which credits have defaulted at time T, correlated random numbers following a standard normal distribution have to be generated for the change in the latent variables X_i and the relative changes in the latent variables have to be compared with the lower thresholds.

1.1.2 Reduced Form or Hazard Rate Approach

In contrast to the structural approach, where the default probability is conditioned explicitly on the value of the regarded firm, reduced form models use actual credit prices to extract the required default probabilities. In the model developed by Duffie and Singleton [6] default events are determined by the hazard rate function $h(t)$. The

hazard rate function gives the instantaneous default probability for a certain time t conditional on no default before time t,

$$h(t) = \mathbb{P}[t < \tau \leq t + \delta t | \tau > t] \quad = \quad \frac{F(t + \delta t) - F(t)}{1 - F(t)} \tag{1.5}$$

$$= \quad \frac{f(t)\delta t}{1 - F(t)} \tag{1.6}$$

$$= \quad -\frac{S'(t)}{S(t)} \tag{1.7}$$

The distribution function $F(t)$ gives the probability that default happens before or at time t. The survival function $S(t)$ gives the probability that default happens after time t. With the equations above the distribution function $F(t) = \mathbb{P}(\tau \leq t)$ of τ (=default time) and the survival function $S(t) = \mathbb{P}(\tau > t) = 1 - F(t)$ can be related to the hazard rate function $h(t)$ as follows:

$$S(t) \quad = \quad e^{-\int_0^t h(s)ds} \tag{1.8}$$

$$F(t) \quad = \quad 1 - e^{-\int_0^t h(s)ds} \tag{1.9}$$

$F(t)$ is the marginal distribution of the default time for a certain credit; this marginal distribution can be computed from market-quoted credit default swap spreads or defaultable bonds [9]. With $F(t)$ given for all credits, the joint distribution function can be derived from the marginal distributions via a copula approach.

1.2   Single-Name Credit Derivatives

In this chapter single-name derivatives - in the sense that there is only one underlying product - are introduced. The name is somewhat misleading as even in a single-name derivative normally there are at least two counterparties involved (e.g. in a credit default swap there is the reference entity of the underlying and the CDS issuing party).

1.2.1   Credit Default Swap

A credit default swap (CDS) is a product which gives the protection buyer the right to demand compensation payments from the protection seller if the underlying defaults. This protection is guaranteed until the maturity T of the credit default swap. In return for the guaranteed protection, the protection buyer has to pay a premium to
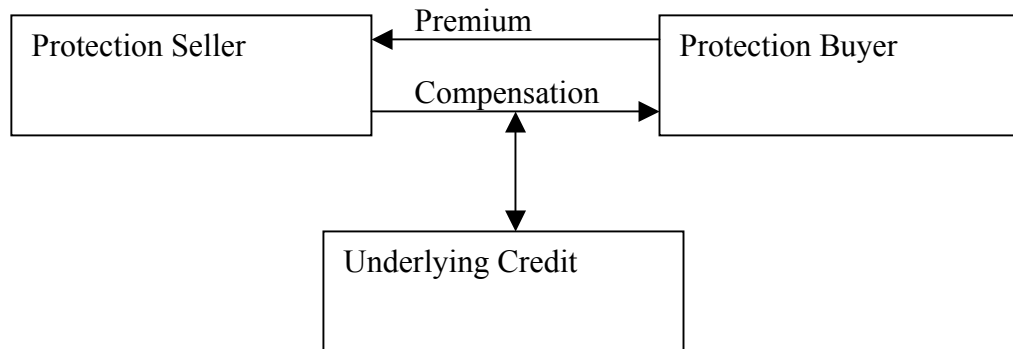
Figure 1.1: Schematic picture of a credit default swap (CDS).

the protection buyer until maturity or until the underlying defaults. This premium payment is called spread payment. The periodic payments of the protection buyer to the protection seller are the fixed side of the swap and denoted as premium leg. The compensation payment of the protection seller to the protection buyer in case of default of the underlying credit is the floating side of the swap and is also called default leg.

To price a credit default swap the following steps have to be accomplished:

- Determine the default probability until maturity.

- Determine the present value of the fixed leg of the CDS (these are the payments the protection buyer pays to the protection seller).

- Determine the present value of the float leg of the CDS (this is the payment the protection buyer receives from the protection seller in case of default).

An elementary way to get the required default probabilities as function of time is to discount the cashflows of market-quoted bonds of the underlying party to today (with the risk free rate). The present value calculated in this way does not enclose the credit risk and is too big compared to the market price. The factors with which the discounted cashflows must be multiplied in order to make their sum equal to the market price are the default probabilities. To get the term structure of credit default probabilities, a procedure which is comparable to bootstrapping coupon bearing bonds to get the zero yield curve (for further information see [9]) has to be accomplished:

- Sort bonds with respect to their maturities from small to big maturities.

- Compute the present value of the first bond by discounting with the risk free rate.

- Find the factor with which the present value has to be multiplied to make the product equal to the market price of the bond.

- Compute the present value of the next bond. Coupons are discounted by multiplying with the risk free rate and the corresponding already determined default probability.

- Find the factor with which the nominal has to be multiplied to make the sum of all coupons and the nominal equal to the market price.

1.2.2   Pricing Credit Default Swaps

A credit default swap guarantees that the protection buyer receives a compensation payment from the protection seller if the underlying security defaults before maturity T. The price of the default protection is the premium s (spread) which has to be paid from the protection buyer to the protection seller periodically.

The following properties are required to compute the present value of the fixed and float leg of credit default swaps:

T	Maturity of the credit default swap
N	Nominal of the credit default swap
t_j	Time of the jth premium payment
$\Delta_{j-1,j}$	Time between payment $j-1$ and payment j in years
τ	Default time of the underlying credit
$\mathbb{P}(\tau \leq t_j)$	Risk neutral probability that underlying defaults before or at time t_j
R	Recovery Rate
$B(0, t_j)$	Discount factor
s	Spread

With the properties given above, the present value of the fixed leg (=premium leg) can be computed by:

$$PV_{Premium} = sN \sum_j B(0, t_j)\Delta_{j-1,j}(1 - \mathbb{P}(\tau \leq t_j)) \tag{1.10}$$

This is the present value of the payments the protection seller receives from the protection buyer. The present value of the payments the protection buyer receives in

case of default is:

$$PV_{Default} = N \int_0^T B(0,t)(1-R)\mathbb{P}(\tau = t)dt \qquad (1.11)$$

As the present values of the two legs have to be equal at initiation of a fair CDS, the spread to be paid on the nominal from the protection buyer to the protection seller is given by:

$$s = \frac{\int_0^T B(0,t)(1-R)\mathbb{P}(\tau = t)dt}{\sum_j B(0,t_j)\Delta_{j-1,j}(1-\mathbb{P}(\tau \le t_j))} \qquad (1.12)$$

1.3 Multi-Name Credit Derivatives

In this section multi-name credit products such as

- Credit default swap[3]

- Basket default swap

- Basket CDS tranche or Collateralized Debt Obligation (CDO)

are introduced.

1.3.1 Credit Default Swap

As already mentioned in the section about single-name credit derivatives, in reality there is nothing like a single-name credit derivative. Even for a credit default swap there are two parties to be taken into account: The first is the party of the referenced underlying, and the second is the issuer of the credit default swap. If the issuer defaults before maturity, the credit default swap is worthless to the protection buyer as he will receive no compensation payments in case of default of the referenced underlying. Thus, to be exact, one has to regard the credit default swap as a credit product with two underlying parties and therefore one cannot neglect the correlation between the party of the referenced underlying and the issuer of the credit default swap. For an exact treatment refer to [10].

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[3]If one pays attention to the fact that the counterparty in the credit default swap is also prone to default, the CDS also has to be regarded as multi-name credit derivative.

1.3.2 Basket Default Swap

A basket default swap (BDS) is a product that, like a plain vanilla credit default swap, guarantees protection against loss in case of default. In contrast to a credit default swap that gives protection against losses in one underlying, a basket credit default swap guarantees protection against losses in all underlying credits that are contained in the basket. There are different kinds of basket default swaps - the most popular one is the 1st-to-default BDS, which gives the protection buyer the right to claim compensation for the losses in the first credit defaulted. Similarly, a 2nd-to-default BDS covers the losses of the second underlying defaulted. In general these products are called k^{th}-to-default basket default swap.
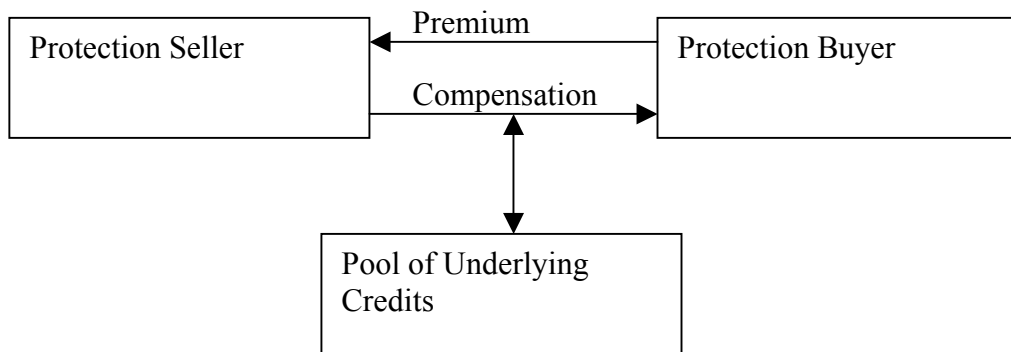
Figure 1.2: Schematic picture of a basket default swap (BDS).

1.3.3 Basket CDS Tranche or CDO

A basket CDS tranche is constructed as slice of a credit portfolio containing a large number n of credits. The basket CDS tranche is defined by its lower boundary a and its upper boundary b. The tranche notional at time t is not affected by a default if the total portfolio loss before default is bigger than b or if the total portfolio after loss is smaller than a. The tranche notional of a basket CDS tranche has the same profile as a put spread (Figure 1.3).

In general there are at least three tranches:

- Equity tranche: This tranche starts from $a = 0$ and covers the very first losses in the portfolio.
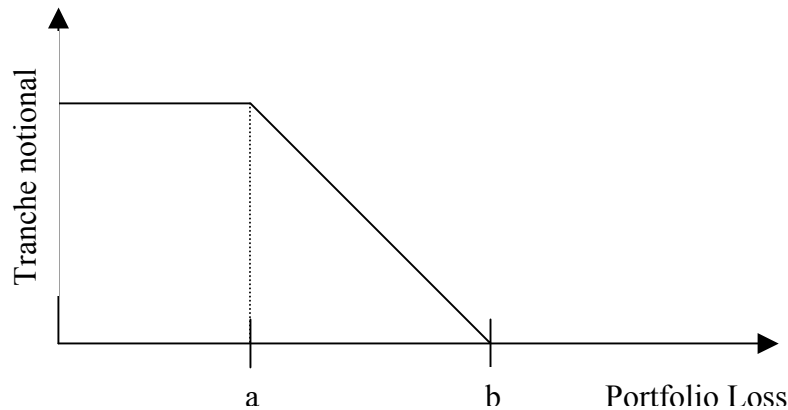
Figure 1.3: Tranche notional of a basket CDS tranche with lower boundary a and upper boundary b.

- Mezzanine tranche: If the portfolio losses have already consumed the complete Equity tranche notional, subsequent defaults are compensated with the Mezzanine tranche notional. Expressed in fractions of the total nominal this tranche is characterized by $a > 0$ and $b < 100\%$.

- Senior tranche: The senior tranche notional is affected by credit defaults only if the notional of all other tranches has already be consumed to cover losses.

The buyer of the tranche can also be called protection seller as he compensates the seller of the tranche for all losses in the tranche (the tranche notional decreases). In return the buyer of the tranche receives periodic payments from the seller of the tranche on the remaining tranche notional.

Like a credit default swap the basket CDS tranche consists of two sides: The regular payments of the protection buyer (tranche seller) are the fixed side. The payment of the protection seller (tranche buyer) in case of default is the float side. For each side there are two ways a default can influence the notional:

- Binary: In case of default the notional decreases with the full amount of the credit. Recovered parts of the credit are not taken into account.

- Recovery: In case of default the decrease in the notional is given by the part of the credit that cannot be recovered.

It is not necessary that the way a default affects the notional is the same for both sides. For example, the effect of a default on the fixed side (premium payments) can be of binary-type (the amount, the fixed side notional decreases, is the notional of

the defaulted credit no matter how much of the credit can be recovered) whereas the effect on the float side (default payment) can be recovery-type. Such a setup would be necessary, for example, if a basket CDS has to be modelled as basket CDS tranche. If such a tranche must be priced, two tranche notionals must be distinguished: The fixed side tranche notional and the float side tranche notional.

Chapter 2

Pricing Multi-Name Credit Derivatives

With the increasing popularity of credit products having a payoff profile depending on a whole credit portfolio modelling correlated defaults has also gained importance. Examples of such products are basket default swaps (k^{th}-to-default swap) and basket CDS tranches or CDOs (Collateralized Debt Obligations). To price such multi-name credit products, it is not suffcient to know the default probabilities for all credits, but it is also necessary to determine the joint distribution of all credits in order to evaluate correlated default effects.

The next sections contain an outline of the theoretical background for pricing multi-name credit derivatives. The key idea of modelling correlated default, the so-called copulae, are described. After the copula concept has been introduced, pricing basket default swaps and basket CDS tranches are discussed in detail. These concepts are used to implement different programs to price k^{th}-to-default basket default swaps and basket CDS tranches. The implementation is described in Chapter 3.

2.1 Copulae

A useful concept for computing joint distribution functions are copulae (a detailed description can be found in [18]). Copulae can be used to generate the joint distribution function of a credit portfolio containing n credits if the marginal distributions of all credits are known. Mathematically, a copula is a function $[0, 1]^n \rightarrow [0, 1]$ with the following properties:

$$C(u_1, u_2, ..., 0, ..., u_n) \quad = \quad 0 \tag{2.1}$$

$$C(1, \ldots, 1, u_k, 1, \ldots, 1) \quad = \quad u_k \tag{2.2}$$

Furthermore, for every $\bar{a}$ and $\bar{b}$ in $[0,1]^n$ such that $a_i \leq b_i$ for all i, the volume of the n-box $[\bar{a}, \bar{b}]$ is $V([\bar{a}, \bar{b}]) \geq 0$.

2.1.1 Sklar's Theorem

The probably most important property of copulae is summarized in Sklar's theorem: Sklar's theorem states that for any n-dimensional distribution function $H(x_1, ..., x_n)$ with marginal distribution functions $F_1(x_1), ..., F_n(x_n)$ there exists a n-dimensional copula, such that

$$H(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)) \tag{2.3}$$

Using different copulae $C_1, C_2, ..$ different joint distribution functions can be generated having the same marginal distribution functions $F_1(x_1), ..., F_n(x_n)$ of the random variables $X_1, \ldots, X_n$.

2.1.2 Gaussian Normal Copula

The most popular copula is the Gaussian normal copula, which models joint distribution as n-dimensional Gaussian normal distribution. For the general case the copula of a n-dimensional distribution is given by

$$C_\Sigma(u_1, \ldots, u_n) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \Sigma}} \int\limits_{-\infty}^{\Phi^{-1}(u_1)} \ldots \int\limits_{-\infty}^{\Phi^{-1}(u_n)} \exp\left(-\frac{1}{2}\bar{x}\Sigma^{-1}\bar{x}\right) dx_1 \ldots dx_n \tag{2.4}$$

where Σ is the linear correlation matrix.

Although often not explicitly mentioned, the Gaussian normal copula is used in many models. For example, in the models of KMV [11] and CreditMetrics [17] the joint distribution is also a multi-variate normal, and if one is familiar with market Value-at-Risk calculations, one recognizes the algorithm given above to be exactly the same as for modelling the evolution of market risk factors.

Equation (2.4) describes the n-dimensional joint distribution for a Gaussian normal copula, but this is not sufficient to create correlated random numbers following the joint distribution of a Gaussian normal copula. These numbers are required for Monte Carlo simulations. An algorithm generating the required random numbers X_i can be formulated as follows:

- Generate the Cholesky decomposition of the correlation matrix $\Sigma = AA^T$;

- Generate vector $\bar{z}$ as m independent standard normally distributed random numbers $z_1, \ldots, z_m \in N(0,1)$;

- Generate vector $\bar{X}$ as $\bar{X} = A\bar{z}$.

Having performed all steps above, correlated random numbers fitting the joint normal distribution can be found in X_i.

2.1.3 Student t Copula

Besides the Gaussian normal copula there are other copulae which can be used to generate joint distribution functions. The Student t distribution is a generalization of the normal distribution in the sense that the Student t distribution contains the Gaussian normal distribution as limit. The Student t distribution contains the additional parameter ν, named degrees of freedom. With $\nu \to \infty$ the Student t distribution converges to the normal distribution. Compared to the normal distribution the Student t distribution shows higher probabilities at the margins[1].

The probability density function of a Student t distribution is given by

$$f_\nu(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \tag{2.5}$$

where $\Gamma(x)$ is the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \tag{2.6}$$

If the Γ function is used with even values of ν, then only the following properties of the Γ function are required:

$$\Gamma(x+1) = x\Gamma(x) \tag{2.7}$$
$$\Gamma(1) = 1 \tag{2.8}$$
$$\Gamma(0.5) = \sqrt{\pi} \tag{2.9}$$

The Student t distribution is

$$t_\nu(x) = \int_{-\infty}^x f_\nu(y)dy \tag{2.10}$$

---

[1]The higher probability mass at the margins is often called "fat tails".

In the case $n = 2$ the Student t copula is given by

$$C_{\nu,\rho_{12}}(u, v) = \int\limits_{-\infty}^{t_\nu^{-1}(u)} \int\limits_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi\sqrt{1 - \rho_{12}^2}} \left(1 + \frac{s^2 - 2\rho_{12}st + t^2}{\nu(1 - \rho_{12}^2)}\right)^{-\frac{\nu+2}{2}} ds\, dt \quad (2.11)$$

where ρ_{12} is the correlation between random variable 1 and random variable 2.

The procedure to generate random numbers fitting a Student t copula is similiar to the generation of standard normally distributed random numbers and comprises the following steps:

- Generate the Cholesky decomposition of the correlation matrix $\Sigma = AA^T$;

- Generate vector $\bar{z}$ as m independent standard normally distributed random numbers $z_1, ..., z_m \in N(0, 1)$;

- Generate one χ_ν^2 distributed random number s;

- Generate vector $\bar{y}$ as $\bar{y} = A\bar{z}$;

- Generate vector $\bar{x}$ as $\bar{x} = \frac{\sqrt{\nu}}{\sqrt{s}}\bar{y}$.

2.1.4 Archimedean Copulae

In order to introduce the Archimedean copula family, a continuous, strictly decreasing and convex function $\varphi(u) : [0, 1] \rightarrow [0, \infty)$ for all $u \in [0, 1]$ is considered. The function φ is called generator of the copula. The pseudo-inverse of φ defined as

$$\varphi^{-1}(z) = \begin{cases} \varphi^{-1}(z) & \text{if} \quad 0 < z \leq \varphi(0) \\ 0 & \text{if} \quad \varphi(0) \leq z < \infty \end{cases}$$

is required if $\varphi(0) = \infty$. If $\varphi(0) \neq \infty$, then φ^{-1} is the ordinary inverse function.

With φ and φ^{-1} the function $C : [0, 1]^n \rightarrow [0, 1]$

$$C(u_1, u_2, \ldots, u_n) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_n)) \quad (2.12)$$

is an Archimedean copula if φ^{-1} is monotone. Examples of the Archimedean copula family are

Gumbel Copula The Gumbel copula has the generator $\varphi(u) = (-\ln(u))^\alpha$ with $\alpha \in (1, \infty)$ and is defined as

$$C(u_1, u_2) = \exp\left\{-[(-\ln(u_1))^\alpha + (-\ln(u_2))^\alpha]^{\frac{1}{\alpha}}\right\} \quad (2.13)$$

Clayton Copula The Clayton copula is defined via the generator $\varphi(u) = u^{-\alpha} - 1$. For $\alpha \in (0, \infty)$ the Clayton copula can be written as

$$C(u_1, u_2) = (u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-\frac{1}{\alpha}} \tag{2.14}$$

Frank Copula For the Frank copula the generator is $\varphi(u) = \ln\left(\frac{\exp(-\alpha u) - 1}{\exp(-\alpha) - 1}\right)$ with $\alpha \in \mathbb{R} \backslash \{0\}$. The Frank copula is

$$C(u_1, u_2) = -\frac{1}{\alpha} \ln\left(1 + \frac{(\exp(-\alpha u_1) - 1)(\exp(-\alpha u_2) - 1)}{\exp(-\alpha) - 1}\right) \tag{2.15}$$

General algorithm to create random numbers for a given copula Embrechts et al. [7] propose an algorithm with which random numbers can be created for any copula. They point out that this algorithm is efficient only under certain conditions (it must be possible to write $C_k^{-1}(\cdot | u_1, \ldots, u_{k-1})$ in closed form). First the k-dimensional marginal distribution of C is defined as

$$C_k(u_1, u_2, \ldots, u_n) = C(u_1, \ldots, u_k, 1, \ldots, 1) \tag{2.16}$$

The random numbers $U_1, \ldots, U_n$ follow the joint distribution C. The conditional distribution of U_k for given $U_1, \ldots, U_{k-1}$ is

$$
\begin{aligned}
C_k(u_k | u_1, \ldots, u_{k-1}) &= \mathbb{P}(U_k \leq u_k | U_1 = u_1, \ldots, U_{k-1} = u_{k-1}) \tag{2.17} \\
&= \frac{\frac{\partial^{k-1} C_k(u_1, \ldots, u_k)}{\partial u_1 \ldots \partial u_{k-1}}}{\frac{\partial^{k-1} C_{k-1}(u_1, \ldots, u_{k-1})}{\partial u_1 \ldots \partial u_{k-1}}} = \frac{C_k^{(k-1)}(u_1, \ldots, u_k)}{C_{k-1}^{(k-1)}(u_1, \ldots, u_{k-1})} \tag{2.18}
\end{aligned}
$$

In order to generate random numbers, the following steps have to be accomplished:

- Create n independent uniform random numbers $v_1, \ldots, v_n \in U(0, 1)$;

- First random number: $u_1 = v_2$;

- Second random number $u_2 \in C_2(u_2 | u_1)$

  with $v_2 = C_2(u_2 | u_1)$ follows $u_2 = C_2^{-1}(v_2 | u_1)$

- Third random number $u_3 \in \ldots$

- n^{th} random number $u_n \in C_n(u_n | u_1, \ldots, u_{n-1})$

  with $v_n = C_n(u_n | u_1, \ldots, u_{n-1})$ follows $u_n = C_n^{-1}(v_n | u_1, \ldots, u_{n-1})$

In case of an Archimedean copula $C_k^{-1}(u_k|u_1,\ldots,u_{k-1})$ can be written in closed form and thus the algorithm described above can be used to generate random numbers. For an Archimedean copula (2.12) the conditional distribution $C_k(u_k|u_1,\ldots,u_{k-1})$ can be formulated as

$$C_k(u_k|u_1,\ldots,u_{k-1}) = \frac{\varphi^{-1(k-1)}(\varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_k))}{\varphi^{-1(k-1)}(\varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_{k-1}))} \tag{2.19}$$

For the Clayton copula with the generator $\varphi(u) = u^{-\alpha} - 1$ and its inverse $\varphi^{-1}(v) = (v+1)^{-\frac{1}{\alpha}}$ the first derivative of φ^{-1} is $\varphi^{-1(1)}(v) = -\frac{1}{\alpha}(v+1)^{-\frac{1}{\alpha}-1}$ and the kth derivative of φ^{-1} is

$$\varphi^{-1(k)}(v) = (-1)^k \frac{(\alpha+1)(\alpha+2)\cdot\ldots\cdot(\alpha+k-1)}{\alpha^k}(v+1)^{-\frac{1}{\alpha}-k} \tag{2.20}$$

According to the above algorithm the following steps have to be performed:

- Generate n independent uniform random variables $(v_1, v_2, \ldots, v_n)$ from $U(0,1)$

- The first random variable is $u_1 = v_1$

- Set $v_2 = C_2(u_2|u_1) = \frac{\varphi^{-1(1)}(c_2)}{\varphi^{-1(1)}(c_1)}$. With $c_1 = \varphi(u_1) = u_1^{-\alpha} - 1$ and $c_2 = \varphi(u_1) + \varphi(u_2) = u_1^{-\alpha} + u_2^{-\alpha} - 2$ this is

$$v_2 = \left(\frac{u_1^{-\alpha} + u_2^{-\alpha} - 1}{u_1^{-\alpha}}\right)^{-\frac{1}{\alpha}-1} \tag{2.21}$$

  This can be solved in u_2 giving

$$u_2 = \left(v_1^{-\alpha}\left(v_2^{-\frac{\alpha}{\alpha+1}} - 1\right) + 1\right)^{-\frac{1}{\alpha}} \tag{2.22}$$

- $\ldots$

- Set $v_n = C_n(u_n|u_1,\ldots,u_{n-1}) = \frac{\varphi^{-1(n-1)}(c_n)}{\varphi^{-1(n-1)}(c_{n-1})}$.

$$v_n = \left(\frac{u_1^{-\alpha} + u_2^{-\alpha} + \ldots + u_{n-1}^{-\alpha} + u_n^{-\alpha} - n + 1}{u_1^{-\alpha} + u_2^{-\alpha} + \ldots + u_{n-1}^{-\alpha} - n + 2}\right)^{-\frac{1}{\alpha}-n+1} \tag{2.23}$$

  This can be solved in u_n giving

$$u_n = \left\{\left(u_1^{-\alpha} + u_2^{-\alpha} + \ldots + u_{n-1}^{-\alpha} - n + 2\right)\left(v_n^{\frac{\alpha}{\alpha(1-n)-1}} - 1\right) + 1\right\}^{-\frac{1}{\alpha}} \tag{2.24}$$

2.1.5 Measure of dependence

Given two random variables X_1 and X_2, there are different measures characterizing the dependence structure between X_1 and X_2. For some distributions, like for example the elliptical distributions (e.g. the Gaussian normal or Student t distribution), the linear correlation coefficient provides a good method to describe dependence. As pointed out by Embrechts et al. [7] for other distributions like the Archimedean distributions the linear coefficient is inappropriate or can even be misleading[2]. For such distributions Kendall's tau and Spearman's rho are alternatives to describe dependence.

Pearson's correlation coefficient ρ The most popular correlation measure is Pearson's linear correlation coefficient. For two random variables X_1 and X_2 with finite variances the linear correlation coefficient is

$$\rho(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}} = \frac{E(X_1, X_2) - E(X_1)E(X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}} \tag{2.25}$$

Kendall's tau   Two observations (x_1, x_2) and (x_1', x_2') of the random vector (X_1, X_2) are concordant if $(x_1 - x_1')(x_2 - x_2') > 0$ and discordant if $(x_1 - x_1')(x_2 - x_2') < 0$. Kendall's tau is defined for the random variables X_1 and X_2 as the probability of concordance minus the probability of discordance:

$$\tau(X_1, X_2) = \mathbb{P}[(X_1 - X_1')(X_2 - X_2') > 0] - \mathbb{P}[(X_1 - X_1')(X_2 - X_2') < 0] \tag{2.26}$$

where (X_1', X_2') is a independent copy of (X_1, X_2).

With the copula $C(u_1, u_2)$ modelling the correlation structure of the random variables X_1 and X_2 this can be written as

$$\tau(X_1, X_2) = 4 \iint_{[0,1]^2} C(u_1, u_2) du_1 du_2 - 1 \tag{2.27}$$

Kendall's tau is equal to -1 if two random variables are countermonotonic, $+1$ if two random variables are comonotonic and 0 if two random variables are independent.

For the Clayton copula Kendall's tau is equal to $\tau = \frac{\alpha}{\alpha+2}$. Thus the parameter α describes the dependence in a Clayton copula correlation structure; increasing α has the same qualitative effect as increasing correlation within the Clayton copula correlation structure.

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[2]An example, how the usage of the linear correlation coefficient for Archimedean copulae results in misleading implications, can be found in [7].

Spearman's rho   Spearman's rho is defined as

$$\frac{\rho_S(X_1, X_2)}{3} = \mathbb{P}[(X_1 - X_1')(X_2 - \bar{X}_2) > 0] - \mathbb{P}[(X_1 - X_1')(X_2 - \bar{X}_2) < 0] \quad (2.28)$$

where (X_1', X_2') and $(\bar{X}_1, \bar{X}_2)$ are independent copies of (X_1, X_2).

2.2   Pricing Basket Default Swaps

A kth-to-default basket default swap gives protection against the kth default in the underlying pool of credits. Because of this, the whole joint distribution of all underlyings must be taken into account. There are two ways of computing the value of a basket default swap:

- Simulate a large number of random scenarios with random numbers following the joint distribution.

- Use a factor model to make semi-explicit computations tractable.

2.2.1   Monte Carlo Method

The present value of a kth-to-default basket depends on the time the kth credit defaults. In order to determine the default time of the kth credit, the default times of all credits in the underlying basket must be known. Pricing a kth-to-default basket default swap thus can be split up into the following steps:

- Generate the correlated default times τ_i for all underlyings in the basket

- Sort the credits with respect to their default time τ_i

- Determine the kth default time τ^k

- Determine the present value of the premium leg

- Determine the present value of the default leg

- Repeat all steps above until the required number of scenarios has been simulated

The subsequent sections give a detailed description of these steps.

Generating the correlated default times   At this point the marginal distribution of the underlying credits as well as the chosen model for the joint distribution become important. In general this step consists of creating random numbers, which are transformed to follow the joint distribution, and computing the default times τ_i out of the random numbers. In order to make things a bit more specific, a Gaussian normal copula is considered. As already described in Section 2.1.2 about Gaussian normal copulae the computation consists of

- generating correlated normal random numbers as in Section 2.1.2

- generating $[0, 1]$ distributed numbers $u_i = \Phi(x_i)$ where $\Phi(x)$ is the normal distribution function

- computing the default times $\tau_i = F_i^{-1}(u_i)$ where $F_i(t) = \mathbb{P}(\tau_i \leq t)$ is the marginal distribution of credit i

With the last two steps the realization x_i of the latent variable X_i is linked to the default probabilities of the underlying credit and the default time τ is computed.[3] The vector $\bar{\tau}$ contains the default times τ_i of all n underlyings in the basket. For a kth-to-default basket default swap the default times have to be brought into an ascending order, and the kth credit defaulting at time τ^k has to be found.

Computing the present value of the premium leg   The payments of the premium leg are the compensation the protection seller receives for taking over the credit risk of the underlying. The premium is paid as long as the underlying credit has not defaulted but not longer than to the maturity of the contract. Whether accrued premium payments between payment dates are taken into account, depends on the contractual agreement. With the kth default time τ^k given, the present value of the premium leg can be computed by

$$PV_{Premium} = sN \sum_j B(0, t_j) \Delta_{j-1,j} \tag{2.29}$$

where N is the nominal of the BDS, t_j are the payment dates of the premium leg $(t_j \leq T)$, s is the percentile amount of the nominal to be paid at payment dates and

---

[3]In the section about Merton's model the equation $D_i = \Phi^{-1}(F_i(t))$ is given: Expressed in terms of the latent variable default occurs if $X_i \leq D_i = \Phi^{-1}(F_i(t))$. Let τ_i be the time the threshold D_i is hit and default occurs. Then the above equation can be reformulated as $\tau_i = F_i^{-1}(\Phi(D_i))$: In this context default occurs if $t \geq \tau_i = F_i^{-1}(\Phi(D_i))$.

$\Delta_{j-1,j}$ is the time between two premium payments:

$$\Delta_{j-1,j} = \left\{ \begin{array}{l} (t_j - t_{j-1})\, 1_{\{\tau^k > t_j\}} \\ (\tau - t_{j-1})\, 1_{\{\tau^k \leq t_j \wedge \tau^k > t_{j-1}\}} \end{array} \right. \quad \text{(accrued premium payment)}$$

If $\tau^k \leq t_j$ and accrued premium payments are agreed, the year fraction $\Delta_{j-1,j}$ is $\tau - t_{j-1}$; if accrued premium payments are not agreed, $\Delta_{j-1,j}$ is 0.

Computing the present value of the default leg With the kth default time τ^k given, the present value of the default leg can be computed with the subsequent equation

$$PV_{Default} = B(0, \tau^k)N(1 - R_k)1_{\{\tau^k \leq T\}} \tag{2.30}$$

where T is the maturity of the BDS, R_k is the recovery rate of the kth defaulted credit and $B(0,t)$ is the discount factor, which gives the present value of one unit paid at time t.

Pricing the Basket Default Swap The term of pricing a basket default swap can be understood in two ways:

- For a given BDS with a fixed spread rate the value of the contract has to be determined. This can be done by calculating the difference between the present values of the default and premium leg. From the protection seller's point of view this can be formulated as

$$PV = \frac{1}{N}\left(\sum_{i=1}^{N} PV_{Premium}(i) - \sum_{i=1}^{N} PV_{Default}(i) \right) \tag{2.31}$$

  where i is the index of the scenario.

- For a given BDS the basket spread rate has to be determined, which makes the BDS worthless for both sides. This can be done by dividing the present value of the default leg through the present value of the premium leg (calculated with $s = 1$ in Equation 2.29).

$$s = \frac{\sum_i PV_{Default}(i)}{\sum_i PV_{Premium}(i)} \tag{2.32}$$

2.2.2 Semi-Explicit Approach

While the Monte Carlo method prices credit products by simulating the default times and calculating the price of the product with respect to the simulated default times in a large number of scenarios, in the semi-explicit approach developed by Laurent and Gregory [12] the probabilities required to compute the present value of the kth-to-default basket CDS, namely the probability $\mathbb{P}(N(t) = m)$ that a certain number of credits has defaulted at time t and the probability $Z_k^i(t)$ that at time t credit i has defaulted as k^{th} credit in the basket, are determined. Given these probabilities, the present value of the premium and default leg can be calculated analytically.

Factor Copulae In order to determine the required default probabilities, Laurent and Gregory use a factor copula approach to model X_i, which is the default determining property of a credit. In the factor model X_i consists of two parts: The first part is a for all credits identical Gaussian random variable V and the second part is a for each credit specific Gaussian random variable V_i:

$$X_i = \beta_i V + V_i \sqrt{1 - \beta_i^2} \tag{2.33}$$

The factor β_i determines how strong X_i is linked to the evolution of the global random variable V. V and V_i are independent standard normally distributed random numbers. Because of this, the correlation between two credits is $\text{Cov}(X_i, X_j) = \beta_i \beta_j$. The link between the X_i and the default time τ_i is given via the marginal distribution of the default time $\mathbb{P}(\tau_i \leq t) = F_i(t)$:

$$X_i = \beta_i V + V_i \sqrt{1 - \beta_i^2} \leq \Phi^{-1}(F_i(t)) \tag{2.34}$$

With the equation above the condition that credit i defaults can be expressed in terms of V_i as

$$V_i \leq \frac{\Phi^{-1}(F_i(t)) - \beta_i V}{\sqrt{1 - \beta_i^2}} \iff \tau_i \leq t \tag{2.35}$$

The conditional default probability $p_t^{i|V}$ that credit i defaults at time t conditional on V is

$$p_t^{i|V} = \mathbb{P}(\tau_i \leq t|V) = \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \beta_i V}{\sqrt{1 - \beta_i^2}} \right) \tag{2.36}$$

Because of the independence of V and V_i the Gaussian factor copula can be written as:

$$C(u_1, \ldots, u_n) = \int \left(\prod_{i=1}^{n} \Phi \left(\frac{\Phi^{-1}(F_i(t)) - \beta_i v}{\sqrt{1 - \beta_i^2}} \right) \right) \varphi(v) dv \qquad (2.37)$$

where $\varphi(v)$ is the normal probability density function of v.

Probability $\mathbb{P}(N(t) = k)$ of having k credits defaulted at time t Let $N(t)$ be the number of defaulted credits at t: $N(t) = \sum_i N_i(t)$ with $N_i(t) = 1_{\{\tau_i \leq t\}}$. The probability $\mathbb{P}(N(t) = k)$ is required to compute the present value of the premium leg of a basket default swap. Furthermore, $\mathbb{P}(N(t) = k)$ is required for a homogeneous basket default swap to compute the present value of the default leg.

Using a probability generating function approach, the probability $\mathbb{P}(N(t) = k)$ that k credits have defaulted at time t can be determined:

$$\psi_{N(t)}(u) = E[u^{N(t)}] = \sum_{k=0}^{n} \mathbb{P}(N(t) = k) u^k \qquad (2.38)$$

This can be written as

$$E[u^{N(t)}] = E[E[u^{N(t)}|V]] = E\left[E\left[\prod_i u^{N_i(t)}|V \right] \right] \qquad (2.39)$$

and if it is taken into account that $E[u^{N_i(t)}|V] = 1 - p_t^{i|V} + u\, p_t^{i|V}$, the probability generating function looks like

$$\psi_{N(t)}(u) = E[u^{N(t)}] = \sum_{k=0}^{n} \mathbb{P}(N(t) = k) u^k = E\left[\prod_{i=1}^{n} \left(1 - p_t^{i|V} + u\, p_t^{i|V} \right) \right] \qquad (2.40)$$

The required probability $\mathbb{P}(N(t) = k)$ can be determined by calculating the coefficient of the u^k term.

Present value of the premium leg If the probability $\mathbb{P}(N(t) = k)$ is known, the present value of the premium leg can be determined. The basket default swaps, regarded by Laurent and Gregory, have the following modalities:

- The premium s is paid on the protected nominal N;

- The premium is paid at certain payment dates t_j;

- For simplification reasons Laurent and Gregory neglect accrued premium payments;

- The time between the payment dates t_{j-1} and t_j is given as year fraction $\Delta_{j-1,j}$;

- $B(0,t)$ is the discount factor which discounts a payment at time t to time 0.

With these assumptions the present value of the premium leg is

$$PV = sN \sum_j \Delta_{j-1,j} B(0,t_j) \mathbb{P}(N(t_j) < k) \qquad (2.41)$$

$$= sN \sum_j \Delta_{j-1,j} B(0,t_j) \sum_{i=0}^{k-1} \mathbb{P}(N(t_j) = i) \qquad (2.42)$$

Determining the default probability $Z_k^i(t)$. In order to evaluate the default leg for an inhomogeneous $(k+1)^{th}$-to-default basket default swaps, the probability that credit i defaults at time t having k credits already defaulted before time t is required for all credits in the basket. This probability is denoted as $Z_k^i(t)$ and can be written with $N_i(t) = 1_{\{\tau_i \le t\}}$ and $N^{(-i)}(t) = \sum_{j \ne i} N_j(t)$ as

$$Z_k^i(t) = \lim_{t' \to t} \frac{1}{t' - t} \mathbb{P}(N_i(t') - N_i(t) = 1, N^{(-i)}(t) = k) \qquad (2.43)$$

$\mathbb{P}(N_i(t') - N_i(t) = 1, N^{(-i)}(t) = k)$ can be computed using the joint probability generating function of $(N_i(t') - N_i(t), N^{(-i)}(t))$ defined by:

$$\psi(u,v) = E\left[u^{N_i(t') - N_i(t)} v^{N^{(-i)}(t)} \right] \qquad (2.44)$$

After some computations (refer to Appendix A), the subsequent equation can be determined, in which the required probability $Z_k^i(t)$ emerges:

$$\sum_{k=1}^{n-1} Z_k^i(t) v^k = E\left[\frac{dp_t^{i|V}}{dt} \prod_{j \ne i} \left(1 - p_t^{j|V} + p_t^{j|V} v \right) \right] \qquad (2.45)$$

The probability that credit i defaults as $(k+1)^{th}$ is given by the coefficient of the v^k term.

Default Leg With the default probability $Z_k^i(t)$ given for all credits in the basket, the present value of the default leg can be determined. The fraction of the nominal which is paid in case of default is given by the recovery rate R_i. As consequence the protection seller has to pay $M_i = N_i(1 - R_i)$ in case of default to the protection buyer. With the properties defined above, the value of the default leg is

$$PV = \int_0^T \sum_{i=1}^n B(0,t) M_i Z_{k-1}^i(t) dt \qquad (2.46)$$

where k is the order of the basket (k^{th}-to-default).

2.3   Pricing Basket CDS Tranches

A basket CDS tranche, as described in detail in Section 1.3.3, gives protection to the protection buyer against losses between the lower boundary a and the upper boundary b. In this section a basket CDS tranche is analyzed from the perspective of the tranche buyer (that is the protection seller). The tranche buyer receives regular payments at times t_j on the notional remaining in the tranche from the tranche seller.

Pricing basket CDS tranches can be accomplished in two ways:

- In Monte Carlo simulations the loss as well as the tranche notional are computed for every scenario individually. With equation (2.51), (2.52) and (2.53) the present value of the tranche is computed.

- In the semi-explicit approach the loss distribution is computed via the characteristic function of the loss. A Fourier Transformation of the characteristic function provides the loss distribution. With the loss distribution the expected tranche notionals of the float and fixed leg can be calculated. If the expected tranche notionals are known, the present value of the basket tranche can be determined.

In the subsequent sections the two techniques to evaluate basket CDS tranches are described in detail.

2.3.1   Monte Carlo Method

In order to get the loss for a credit portfolio containing n credits, the default time τ_i must be determined for all n credits. As the manner a default influences the tranche notional can be different for fixed and float side, two portfolio losses are distinguished. Let $L_{fixed}(t)$ be the portfolio loss for the fixed side at time t and $L_{float}(t)$ the portfolio loss for the float side.

$$L_{fixed}(t) = \sum_{i=1}^{n} N_i(1 - R_i\,1_{\{fixed\,type=recovery\}})\,1_{\{\tau_i \leq t\}} \qquad (2.47)$$

$$L_{float}(t) = \sum_{i=1}^{n} N_i(1 - R_i\,1_{\{float\,type=recovery\}})\,1_{\{\tau_i \leq t\}} \qquad (2.48)$$

In the equations above N_i is the nominal of credit i, R_i is the recovery rate of credit i and τ_i is the default time of credit i. With a lower boundary a and an upper boundary

b the tranche notional is

$$N_{fixed\,side}(t) = \max(b - L_{fixed}(t), 0) - \max(a - L_{fixed}(t), 0) \qquad (2.49)$$

$$N_{float\,side}(t) = \max(b - L_{float}(t), 0) - \max(a - L_{float}(t), 0) \qquad (2.50)$$

The present value of the fixed side payments can be written as

$$PV_{fixed\,side} = s \sum_{j=1}^{m} \Delta_{j-1,j} B(0, t_j) N_{fixed}(t_j) \qquad (2.51)$$

where $t_m = T$ is the maturity of the basket CDS tranche, $\Delta_{j-1,j}$ is the year fraction between two payments, $B(0, t_j)$ is the discount factor and s is the percentile premium the protection seller receives.

If it is assumed that compensation payments for default are paid in certain intervals only, the present value of the float side payments is

$$PV_{float\,side} = \sum_{j=1}^{l} (N_{float}(t_{j-1}) - N_{float}(t_j)) B(0, t_j) \qquad (2.52)$$

The present value of the tranche (from the protection seller's point of view) is the difference between the two present values computed above:

$$PV_{Tranche} = PV_{fixed} - PV_{float} \qquad (2.53)$$

Performing all described steps N times, adding up all present values and dividing the sum by N gives the expected value of the tranche.

2.3.2 Semi-Explicit Approach

While in the Monte Carlo method the portfolio loss (and thus also the present value of the tranche) is computed for each scenario with respect to the simulated default times τ_i individually, and the expected value of the basket CDS tranche is the average of the present values, in the semi-explicit approach the whole loss distribution $\mathbb{P}(L(t) = k)$ is used to determine the tranche spread.

The first step in the semi-explicit approach is to determine the loss distribution $\mathbb{P}(L(t) = k)$. This is accomplished by the characteristic function of $L(t)$:

$$
\begin{aligned}
\Psi_{L(t)}(u) &= E[e^{iL(t)u}] \qquad (2.54)\\
&= E\left[\prod_{j=1}^{n} \left(\left(1 - p_t^{j|V}\right) + p_t^{j|V} \cdot e^{iM_j u}\right)\right]\\
&= \int_{-\infty}^{+\infty} \varphi(V) \prod_{j=1}^{n} \left(\left(1 - p_t^{j|V}\right) + p_t^{j|V} \cdot e^{iM_j u}\right) dV
\end{aligned}
$$

with the loss M_j on credit j in case of default (this is $N_j(1 - R_j)$ for recovery type or N_j for binary type) and $\varphi(V)$ the normal probability density function for the global latent variable V. The characteristic function (2.54) can be expressed in terms of the Fourier Transform of $\phi(k)$:

$$\Psi_{L(t)}(u) = \int_{-\infty}^{\infty} e^{iku}\phi(k)dk \tag{2.55}$$

The interesting point is that the probability $P(L(t) = k)$ is equal to the Fourier Transform $P(L(t) = k) = \phi(k)$. The required loss distribution for a certain time t can be determined by Fourier Transformation of the characteristic function $E[e^{iL(t)u}]$:

$$\phi(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iku}\Psi_{L(t)}(u)du \tag{2.56}$$

In practice the first step to determine the probabilities $\mathbb{P}(L(t) = k)$ is to calculate the characteristic function with equation (2.54) then to make a Fourier Transformation of the characteristic function as described in equation (2.55) to get $\phi(k)$.

The second step is to determine what impact a certain portfolio loss k has on the tranche notional. The connection between portfolio loss and tranche notional $N(k)$ is displayed in Figure 1.3 and can mathematically be stated as

$$N(k) = (b - a) + (a - k)1_{\{k \geq a\}} + (k - b)1_{\{k \geq b\}} \tag{2.57}$$

This function maps a certain portfolio loss to the corresponding tranche notional. If the leg types of fixed and float leg differ, two portfolio losses have to be determined in the first step and in the second step the two portfolio losses have to be mapped to two tranche notionals with equation (2.57).

With the loss distribution determined and the mapping function $N(k)$ between portfolio loss and tranche notional the expected tranche notional is

$$N = E[N] = \sum_{k=0}^{+\infty} N(k)\mathbb{P}(L(t) = k) \tag{2.58}$$

If the expected tranche notional has been computed, equation (2.51) and (2.52) provide the present value of the basket tranche.

Chapter 3

Implementation

In this chapter the implementation of the different programs is described in detail. The chapter starts with the definition of reference results which are used to check the implemented programs. Then the programs using Monte Carlo simulations are described. The chapter ends with the description of the programs using the semi-explicit approach to price multi-name credit derivatives.

3.1 Benchmark Results - Reference Case

In order to verify whether the results in the implemented programs are reasonable, benchmark results are required.

Schmidt and Ward published an article [21] in which they investigate a kth-to-default basket with three underlying credits. They assume that the spread of credit default swaps on the three underlyings are 0.9%, 1.0% and 1.1% for maturities 1 to 5 years. The recovery rate and the linear correlation of the underlyings is 20% and 50%. This credit basket is called reference case throughout this thesis. The results of the reference case are shown in Table 3.1. The first step to check the

Maturity (yrs)	1st	2nd	3rd
1	2.63	0.34	0.04
2	2.56	0.42	0.06
3	2.51	0.47	0.08
4	2.47	0.51	0.09
5	2.44	0.55	0.10

Table 3.1: Benchmark basket spreads from Schmidt and Ward [21].

implementation with Schmidt and Ward is to determine default probabilities for the

underlyings which give the spreads mentioned above for the CDS. The spread of a credit default swap is given by equation (1.12). As the CDS used by Schmidt and Ward have the same spread for all maturities, determining the default probabilities is done by a procedure similar to bootstrapping. Firstly, the default probability for one year is determined by changing it as long as the resulting spread does not agree with the required spread. Then with the one year default probability determined, the two year default probability is changed until the resulting spread agrees with the required one. This procedure is done for all maturities and for all three CDS in an auxiliary Excel spreadsheet. The resulting default probabilities are shown in the following table:

	Credit 1	Credit 2	Credit 3
Spread	0.9%	1.0%	1.1%
Maturity			
0	0.00000	0.00000	0.00000
1	0.01097	0.01217	0.01337
2	0.02172	0.02417	0.02653
3	0.03249	0.03600	0.03950
4	0.04304	0.04768	0.05229
5	0.05346	0.06519	0.06487
6	0.06380	0.07056	0.07730

Table 3.2: Default probabilities for the reference case.

3.2 Monte Carlo Method

General Computations Starting point for the Monte Carlo evaluation of multi-name credit products are the correlated default times τ_i.

For a Gaussian normal copula correlation structure correlated $N(0,1)$-distributed random numbers are required. These numbers are generated by two algorithms. $N(0,1)$ distributed random numbers are created according to Marsaglia [5] and the uncorrelated random numbers are transformed into correlated ones (with the correlation matrix Σ) by multiplying them with the Cholesky decomposed matrix A (the relationship between Σ and A is $\Sigma = AA^T$).

For a Student t copula correlation structure the correlated $N(0,1)$-distributed random numbers are multiplied additionally with $\sqrt{\nu/s}$ where ν is the degree of freedom of the t distribution and s is a χ^2_ν-distributed random number.

The correlated random numbers are transformed into $[0, 1]$-distributed numbers (these are the default probabilities):

- $u_i = \Phi(x_i)$ for Gaussian normal copula

- $u_i = t_\nu(y_i)$ for Student t copula $(y_i = \sqrt{\nu/s}\, x_i)$

For the Clayton copula the generation of random numbers is accomplished with the algorithm described in Section 2.1.4.

With the $[0, 1]$-distributed numbers, the default times τ_i are determined by $\tau_i = F_i^{-1}(u_i)$ where $F_i(t)$ is the marginal default probability of underlying i.

3.2.1 Implementation

An overview for which products and copulae a Monte Carlo pricing program has been implemented is given in Table 3.3:

Copula Product	Gaussian normal	Student t	Clayton
k^{th}-to-default	Program 1	Program 3	Program 5
Basket CDS Tranche	Program 2	Program 4	Program 6

Table 3.3: Matrix of products and copulae for which Monte Carlo pricing programs are implemented.

Progam 1 Monte Carlo simulation of an inhomogeneous k^{th}-to-default basket default swap. Joint default is modelled as **Gaussian normal copula**. After the correlated default times τ_i have been simulated, the value of the premium and the default leg are determined with equations (2.29) and (2.30). With the present value of the default and premium leg for all scenarios the spread of the BDS is determined as quotient of the sum of the present values of the default legs divided by the sum of the present values of the premium legs (equation (2.32)).

Program 2 Monte Carlo simulation of an inhomogeneous basket CDS tranche. Joint default is modelled as **Gaussian normal copula**. With the correlated default times τ_i given, the loss of the float (N_{float} as in (2.48)) and the fixed side (N_{fixed} as in equation (2.47)) of the tranche can be determined at any time. Equation (2.51) and (2.52) provide the present value of the fixed and float side.

Program 3 Monte Carlo simulation of an inhomogeneous k^{th}-to-default basket default swap. Joint default is modelled as **Student t copula**. The Student t distribution is given as integral over the Student t probability function from $-\infty$ to x:

$$t_\nu(x) = \int\limits_{-\infty}^{x} f_\nu(y)dy \tag{3.1}$$

Calculating the integral using Simpson's rule would be far too slow therefore integration is accomplished with the Gaussian quadrature method.[1] After the correlated default times are determined, the spread of the BDS is computed as in Program 1.

Program 4 Monte Carlo simulation of an inhomogeneous basket CDS tranche. In contrast to Program 2 joint default is modelled as **Student t copula**; all other steps are equal to Program 2.

Program 5 Monte Carlo simulation of an inhomogeneous k^{th}-to-default basket default swap. Joint default is modelled as **Clayton copula**. With exception of the copula used this program is identical to Program 1.

Program 6 Monte Carlo simulation of an inhomogeneous basket CDS tranche. Joint default is modelled as **Clayton copula**. With exception of the copula used this program is identical to Program 2.

3.2.2 Comparison with Reference Case

In order to check the implementation of the programs, every program is parameterized to compute spreads for the reference case (see Section 3.1):

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[1] In the Gaussian quadrature framework the integral $\int_a^b f(x)dx$ is computed as sum of N function values $f(x_i)$ at fixed abscissae $x_i \in (a, b)$, where each function value is multiplied with a for the abscissa x_i specific weight $w(x_i)$:

$$\int\limits_{a}^{b} f(x)dx = \sum_{i=1}^{N} w(x_i)f(x_i) \tag{3.2}$$

The abscissae x_i as well as the weights $w(x_i)$ do not depend on the function $f(x)$ to be integrated. An algorithm providing the abscissae x_i and the weights $w(x_i)$ is given in [19].

Program 1 *Normal copula, k^{th}-to-default basket:* A basket CDS consisting of three credits with marginal default probabilities that give CDS spreads of 0.9%, 1.0% and 1.1% is examined. Recovery is set to 20% and linear correlation coefficients to 50%. In each simulation run $N = 5 \cdot 10^6$ scenarios are created to compute the basket spread. Table 3.4 shows the results of the calculations.

Program 2 *Normal copula, basket CDS tranche:* In order to map the reference case as basket CDS tranche, a portfolio consisting of the three credits mentioned above is set up. The premium leg type is set to binary while the default leg type is set to recovery. Lower and upper boundary of the tranche are chosen automatically in such a way that the relevant tranche notional is equivalent to a kth-to-default (e.g. for the 2nd-to-default case with a nominal of 300 and a recovery rate of 20% the lower and upper boundary for the premium tranche notional are set to 100 and 200 whereas the boundaries for the default tranche notional are set to 80 and 160).

Maturity	Schmidt & Ward			k^{th}-basket			Basket Tranche		
	1st	2nd	3rd	1st	2nd	3rd	1st	2nd	3rd
1	2.63	0.34	0.04	2.647	0.331	0.039	2.635	0.330	0.039
2	2.56	0.42	0.06	2.558	0.412	0.060	2.546	0.407	0.061
3	2.51	0.47	0.08	2.508	0.468	0.077	2.500	0.460	0.076
4	2.47	0.51	0.09	2.463	0.509	0.092	2.451	0.501	0.090
5	2.44	0.55	0.10	2.437	0.542	0.104	2.424	0.534	0.102

Table 3.4: Comparison of kth-to-default basket (Program 1) and basket CDS tranche (Program 2) with Schmidt & Ward [21]. Correlation structure is modelled with Gaussian normal copula. The relative errors of the basket spreads (defined as standard deviation of the basket spread divided by the basket spread) are found to be of equal size for Program 1 and Program 2: The relative errors are smaller than 0.7% for 1st-to-default, smaller than 1.8% for 2nd-to-default and smaller than 5.3% for 3rd-to-default.

Program 3 *Student t copula, k^{th}-to-default basket:* In the limit of $\nu \rightarrow \infty$ the Student t distribution is equal to the Gaussian normal distribution. To Compare values with the reference case, very large values of ν have to be regarded. The results displayed in Table 3.5 have been determined with $\nu = 1000$.

Program 4 *Student t copula, basket CDS tranche:* The reference case is mapped as basket tranche as already described. In order to compare the results of the Student t copula with those computed with a normal copula, the parameter ν is set to $\nu = 1000$.

As can be seen in Table 3.4 and 3.5, there are only little differences between the results of the implemented programs and the reference results provided by Schmidt and Ward.

Maturity	Schmidt & Ward			k^{th}-basket			Basket Tranche		
	1st	2nd	3rd	1st	2nd	3rd	1st	2nd	3rd
1	2.63	0.34	0.04	2.63	0.34	0.04	2.63	0.33	0.04
2	2.56	0.42	0.06	2.56	0.41	0.06	2.58	0.42	0.06
3	2.51	0.47	0.08	2.50	0.47	0.08	2.51	0.47	0.08
4	2.47	0.51	0.09	2.46	0.51	0.09	2.48	0.51	0.09
5	2.44	0.55	0.10	2.43	0.55	0.10	2.46	0.54	0.10

Table 3.5: Comparison of kth-to-default basket (Program 3) and basket CDS tranche (Program 4) with Schmidt & Ward [21]. Correlation structure is modelled with Student t copula. The parameter degree of freedom is set to $\nu = 1000$. The relative errors of the basket spreads (defined as standard deviation divided by the basket spread) are found to be of equal size for Program 3 and Program 4: The relative errors are smaller than 0.7% for 1st-to-default, smaller than 1.9% for 2nd-to-default and smaller than 5.4% for 3rd-to-default.

3.2.3 Convergence of Monte Carlo Method

In order to get an impression how fast the Monte Carlo method converges, a Monte Carlo simulation[2] with 30 Mio. scenarios is performed. The error in scenario i is:

$$\text{Error}_i = \frac{\text{Spread}(i_{max}) - \text{Spread}(i)}{\text{Spread}(i_{max})} * 100 \tag{3.3}$$

Figure 3.1 shows the relative error as function of the scenario. As can be seen in the figure, the biggest relative error is about 0.35%, and after 4 Mio. simulations the relative error stays smaller than 0.1%.

3.3 Semi-Explicit Approach

In the semi-explicit approach a factor copula model is used to reduce the complexity in correlation structure. This approach is implemented for a Gaussian normal copula only. The programs implemented in this approach are:

- Program 7: This program prices inhomogeneous k^{th}-to-default basket CDS with a Gaussian normal copula correlation structure as described in Section 2.2.2.

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[2]The simulation computes the spread of the 1st-to-default for Schmidt & Ward's reference basket where the contract has a 5 year maturity.
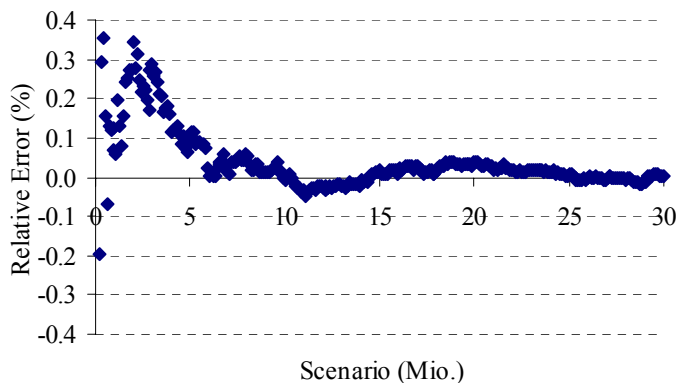
Figure 3.1: This figure shows the relative error in the tranche spread as function of the number of simulation runs.

- Program 8: This program prices inhomogeneous basket CDS tranches with a Gaussian normal copula correlation structure via the characteristic function of the loss (Section 2.3.2).

3.3.1 Semi-Explicit Approach: Basket CDS

The first step is to implement a function which computes the conditional default probability as given in equation (2.36). With $p_t^{i|V}$ the probability $\mathbb{P}(N(t) = k)$, that the number of defaulted credits $N(t)$ at time t is equal to k, can be determined. This can be achieved by directly calculating the coefficient of the u^k term in equation (2.40). The expectation has to be taken with respect to V. The integral is computed with Simpson's-rule, that is the interval $b - a$ is divided into n parts. The integral is given by

$$\int_a^b f(x)dx = \sum_{i=1}^n \left(f(x_i) + 4 f\left(\frac{x_i + x_{i+1}}{2} \right) + f(x_{i+1}) \right) \frac{x_{i+1} - x_i}{6} \qquad (3.4)$$

where $x_1 = 1, x_{i+1} = x_i + \frac{b-a}{n}$ and $x_n = b$.

With the number of defaulted credits known, the present value of the premium leg can already be computed with (2.41). In order to compute the value of the default leg, the probability $Z_k^i(t)$, that credit i defaults at time t after k credits already have defaulted, is required. Equation (2.45) provides $Z_k^i(t)$ as coefficient of the v_k-term. Expanding the product in (2.45) and determining the coefficient of the v^k-term delivers the probability $Z_k^i(t)$.[3] If the probabilities $Z_k^i(t)$ are calculated for the relevant

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[3]The implementation showed that $Z_k^i(t)$ is extremely sensitive to errors in the inverse normal distribution. A first approximation of the inverse normal distribution, which had a relative error less that 10^{-6}, led to errors in the spread bigger than 10%. The final version used to calculate the inverse normal distribution accomplishes a rational approximation with a relative error less than 10^{-8} and uses this result for the Halley method [1] to get an accurate result.

k for all credits i and all relevant times t, the present value of the default leg can be computed with equation (2.46).

This method is used for the calculation of the spread of kth-to-default baskets only. For basket CDS tranches expanding the probability generating function $\Psi_{L(t)} = E[u^{L(t)}]$, like it is done for a kth-to-default in (2.45), is not very useful, as the complete loss distribution is needed and not only one coefficient as for a kth-to-default basket.

3.3.2 Semi-Explicit Approach: Basket CDS Tranche

In contrast to a kth-to-default basket, where simply the probability that credit i defaults as k^{th} credit is needed to price the default leg, for a basket CDS tranche the complete loss distribution of the portfolio is required for valuation. Thus the main focus is to determine the loss distribution of the portfolio, that is the probability that the portfolio is hit by a certain loss as function of the loss $\mathbb{P}(L(t) = x)$. The sought function $\mathbb{P}(L(t) = x)$ can be determined as Fourier Transform[4] of the characteristic function of the portfolio loss $\Psi_{L(t)}(u) = E[e^{iL(t)u}]$. The integration is done with Gaussian quadrature method.

3.3.3 Comparison with Reference Case

k^{th}-**to-default Basket**   In the following table the results of the Semi-Explicit approach are compared with the Schmidt & Ward case:

Maturity (yrs)	Schmidt & Ward			Semi-Explicit		
	1st	2nd	3rd	1st	2nd	3rd
1	2.63	0.34	0.04	2.68	0.33	0.04
2	2.56	0.42	0.06	2.59	0.41	0.06
3	2.51	0.47	0.08	2.54	0.46	0.08
4	2.47	0.51	0.09	2.50	0.51	0.09
5	2.44	0.55	0.10	2.46	0.54	0.10

Table 3.6: Comparison of the semi-explicit approach with Schmidt & Ward.

Comparing the numbers above, one has to keep in mind that Schmidt and Ward take the accrued premium payments into account whereas in Laurent and Gregory's semi-explicit approach these payments are neglected. Because of this, the difference in the case of a 1st-to-default basket with maturity $T = 1$ between 2.63 and 2.68 is not that surprising. A more useful indicator of the correctness of the implementation

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[4] The Fourier Transformation is done as FFT (Fast Fourier Transformation) with the algorithm given in [19].

Maturity (yrs)	Monte Carlo			Semi-Explicit		
	1st	2nd	3rd	1st	2nd	3rd
1	2.6797	0.3312	0.0395	2.6779	0.3317	0.0403
2	2.5862	0.4123	0.0601	2.5928	0.4139	0.0608
3	2.5362	0.4685	0.0765	2.5375	0.4688	0.0772
4	2.4925	0.5101	0.0916	2.4960	0.5103	0.0914
5	2.4660	0.5435	0.1037	2.4631	0.5437	0.1039

Table 3.7: Comparison between the results of the semi-explicit approach and the Monte Carlo results. In this special implementation accrued premium payments are not considered. The relative errors of the basket spreads (standard deviation of the basket spread divided by the basket spread) are smaller than 0.7% for 1st-to-default, smaller than 1.8% for 2nd-to-default and smaller than 5.3% for 3rd-to-default.

is the difference between the numbers computed in the semi-explicit approach and those that are computed via a special Monte Carlo implementation, where the accrued premium payments are also not considered. As can be seen in Table 3.7, there is quite a good congruence between the results in the Monte Carlo approach and in the semi-explicit approach.

Basket CDS tranche To check the implementation of the pricing of basket CDS tranches, Program 8 is parameterized such that the reference case can be described as a basket CDS tranche. In order to do so, the fixed leg type is set to binary and the float leg type is set to recovery. The results of this computation can be seen in the subsequent Table 3.8.

Maturity (yrs)	Schmidt & Ward			Semi-Explicit Tranche		
	1st	2nd	3rd	1st	2nd	3rd
1	2.63	0.34	0.04	2.639	0.328	0.040
2	2.56	0.42	0.06	2.550	0.407	0.060
3	2.51	0.47	0.08	2.497	0.461	0.076
4	2.47	0.51	0.09	2.455	0.501	0.090
5	2.44	0.55	0.10	2.421	0.534	0.102

Table 3.8: Comparison between the results of the semi-explicit approach and Schmidt and Ward's results if the basket CDS is modelled as homogeneous basket CDS tranche.

3.3.4 Bimodal and binomial distribution

For certain credit portfolios the loss distribution can be computed analytically in a simple manner. This is the case if:

- all credits have the same nominal N;

- all credits have the same default probability p;

- all credits have the same recovery rate of $R = 0$;

- the correlation between all credits is the same;

- the correlation between the credits is $\rho = 0$ or $\rho = 1$.

In the $\rho = 1$ case the loss distribution is bimodal; that is, there are only two values populated: Either all credits have defaulted, so that the portfolio loss is 100%, or no credit has defaulted, so that the portfolio loss is 0%. The probability of all credits having defaulted at time t is p and the probability of no credit having defaulted is $1 - p$. This situation is shown in Figure 3.2 for $p = 10\%$. In the $\rho = 0$ case the loss
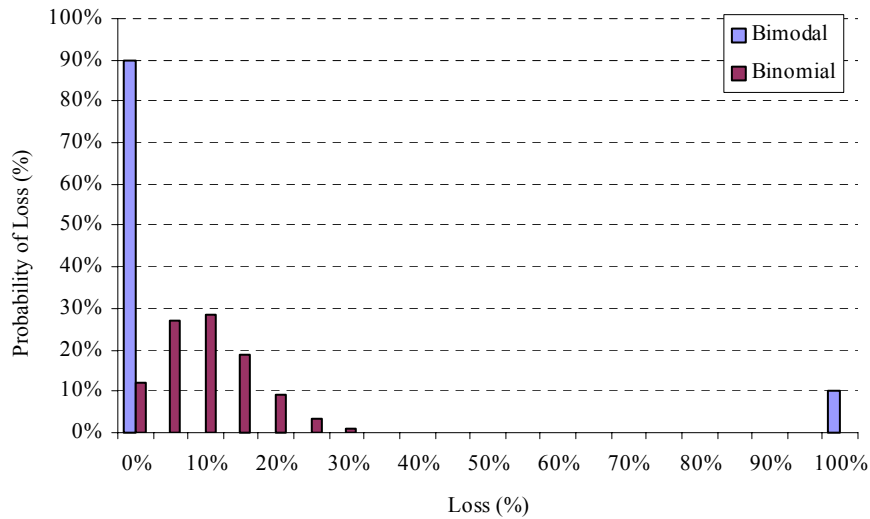
Figure 3.2: This figure shows the relative error in the tranche spread as function of the number of simulation runs.

distribution is binomial:

$$\mathbb{P}(\mathrm{Loss}(t) = N\,k) = N\binom{n}{k}p^k(1-p)^{n-k} \tag{3.5}$$

Figure 3.2 also shows the binomial distribution for a default probability of $p = 10\%$ and $n = 20$. This special setup of a credit portfolio can be used to check whether the

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implementation of Program 8 delivers the expected loss distributions. Hence, Program 8 is parameterized as described above, and the loss distribution is determined and compared with the theoretical expectation (as shown in Figure 3.2). There are no differences to the theoretical values.

3.3.5 Runtime Comparison

The big advantage of the semi-explicit approach is its ability to provide results much faster than Monte Carlo computations. In order to give an impression how big the differences in the required runtimes implied by the chosen method are, the basket spreads are calculated with maturities $T = 1, ..., 5$ for the 1st-to-default basket reference case and the observed runtimes are shown in Table 3.9.

The basket spreads are calculated via:

- Program 1 - Direct Monte Carlo: $5 \cdot 10^6$ scenarios;

- Program 2 - Monte Carlo Tranche: $5 \cdot 10^6$ scenarios, 1st-to-default is modelled as basket CDS tranche;

- Program 8 - Semi-Explicit Approach: 1st-to-default is modelled as basket CDS tranche, integration in equation (2.54) is done with Simpson's rule;

- Program 8 - Semi-Explicit Approach: 1st-to-default is modelled as basket CDS tranche, integration in equation (2.54) is done with a 20-point Gaussian Legendre quadrature method[5];

Table 3.9 shows the observed runtimes[6].

	Monte Carlo Direct	Monte Carlo Tranche	Semi-Explicit Simpson's rule	Semi-Explicit Gaussian Quadrature
Runtime(s)	210	321	5	1

Table 3.9: Observed runtime to compute five 1st-to-default basket spreads for maturities $T = 1, 2, 3, 4, 5$.

It is obvious that the semi-explicit approach is able to provide results much faster than Monte Carlo methods.

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[5]An implementation of a 10-point Gaussian integration algorithm can be found in [19] (see Chapter 4.5).

[6]The calculations are performed on a 600 MHz Intel Pentium III PC with 64 MByte memory.

Chapter 4

Results

In this chapter the results computed with the implemented programs are presented. Section 4.1 gives the results for the reference credit basket, and in Section 4.2 the results for a real world basket CDS tranche are shown.

4.1 Results for the Reference Case

In this section Schmidt and Ward's reference basket [21] is subject of examination.

4.1.1 Gaussian Normal Copula - Basket Spread as Function of the Correlations

For Gaussian normal copulae linear correlation is an important parameter. In this section the influence of ρ on the basket spread is investigated. Changing correlation
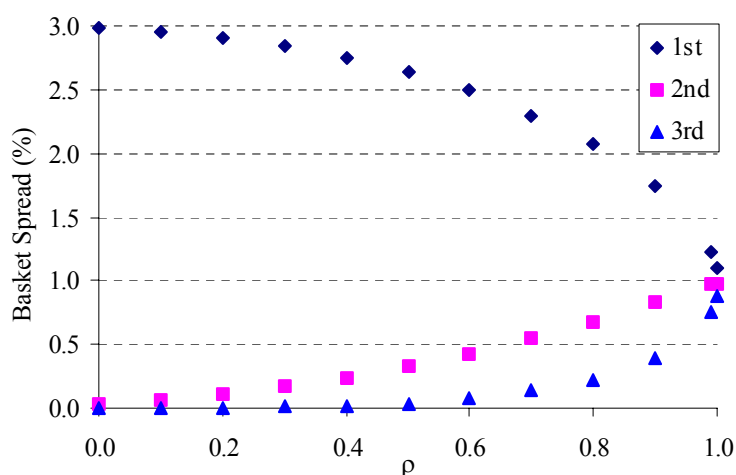
Figure 4.1: Basket spread as function of the linear correlation coefficient ρ. The number of Monte Carlo simulation runs is $N = 5 \cdot 10^6$.

between the credits has a big influence on the basket spread of a kth-to-default basket. Figure 4.1 displays the spread[1] of a 1st-, a 2nd- and a 3rd-to-default basket as function of the correlation for the reference case [21]. With increasing correlation the spread of the 1st-to-default decreases whereas the spread of the 2nd- and 3rd-to-default increases with increasing correlation. The explanation for this dependence is that with increasing correlation the probability for a second or third default increases whereas the probability for a first default decreases. Therefore the spread of the 1st-to-default decreases while 2nd- and 3rd-to-default protection gets more expensive.

4.1.2 Student t Copula - Influence of ν on the Basket Spread

If the correlation structure is modelled as Student t copula, the additional parameter ν influences the spread of a kth-to-default basket. The dependence of the 1st-to-default[2] for the reference case is displayed in the upper left graph of Figure 4.2. With increasing ν the spread increases also and converges to the normal copula spread. As can be seen in the figure the difference between the spread computed with the Student t copula and the spread computed with the Gaussian normal copula of 2.63% is already for $\nu = 64$ very small (about 1% difference). For small values of ν the probability of correlated defaults increases leading to a decrease in the spread. The right upper graph of Figure 4.2 displays the influence of ν on the spread of the 2nd-to-default for the reference case and the lower left graph the dependence of the 3rd-to-default basket spread on the parameter ν.

What can be seen clearly in Figure 4.2 is that the parameter ν of the Student t copula has a dramatic influence on the basket spread.

The results are summarized in the lower right graph of figure 4.2, which shows the relative differences of the Student t basket spread to the Gaussian normal copula basket spread as function of ν. The maximal change is -23% for the 1st-to-default, 115% for the 2nd-to-default and 448% for the 3rd-to-default. This shows clearly what impact the copula used for pricing has on the determined prices.

4.1.3 Clayton Copula - Influence of α on the Basket Spread

The last copula investigated for the reference case is an example of the Archimedean copula family - the Clayton copula. The results are computed with a Monte Carlo simulation as described for the Clayton copula in Section 2.1.4. Figure 4.3 shows the

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[1]These results have been computed with Program 1.

[2]The results are computed with Program 4 for $T = 1$. The number of Monte Carlo simulation runs is set to $N = 5 \cdot 10^6$.
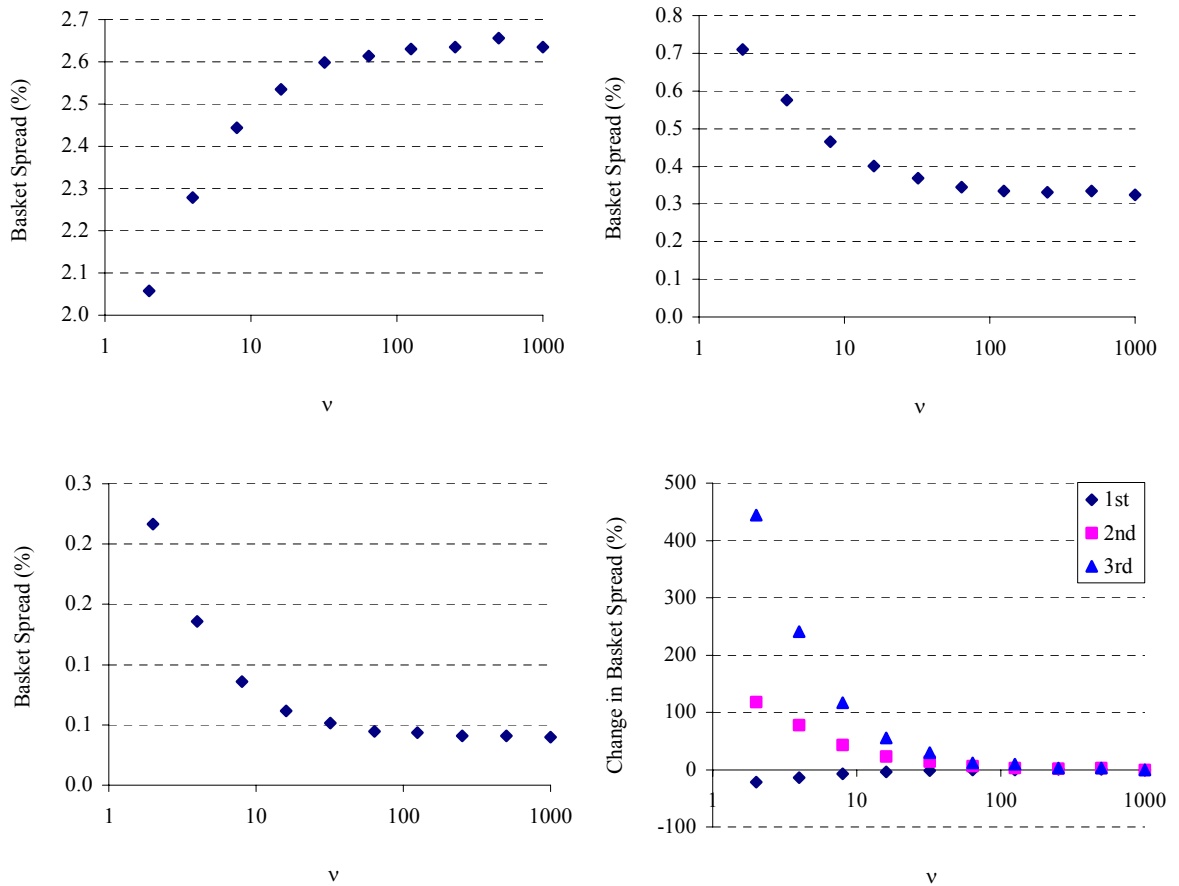
Figure 4.2: Basket Spread as function of the parameter ν; the left upper graph shows the 1st-to-default basket spread - the relative errors (standard deviation divided by basket spread) are smaller than 1.0%; the 2nd-to-default basket spread is displayed in the right upper graph - the relative errors are smaller than 2.9%; the right lower graph shows the 3rd-to-default - the relative errors are smaller than 5.5%; the right lower graph relates the Student t basket spreads to the Gaussian normal basket spreads.

basket spread for maturity $T = 1$ for the 1st-, 2nd- and 3rd-to-default as function of the parameter α. The parameter α is related to the "correlation" within the Clayton copula correlation structure. Kendall's tau as measure of dependence is $\tau = \frac{\alpha}{\alpha+2}$. Thus increasing α is analogous to increasing correlation. For the 1st-to-default basket increasing α results in a decrease of the basket spread whereas for the 2nd- and 3rd-to-default a decrease leads to an increase of the basket spread. The influence α has on the basket spread thus can be compared to the influence the parameter ρ has on the basket spread for the Gaussian normal copula. The form of the curves in Figure 4.1 and 4.3 is different but the qualitative influence of increasing correlation is in both figures the same: An increase in correlation leads to a decrease in the 1st-to-default basket spread and increases in the 2nd and 3rd-to-default basket spreads.
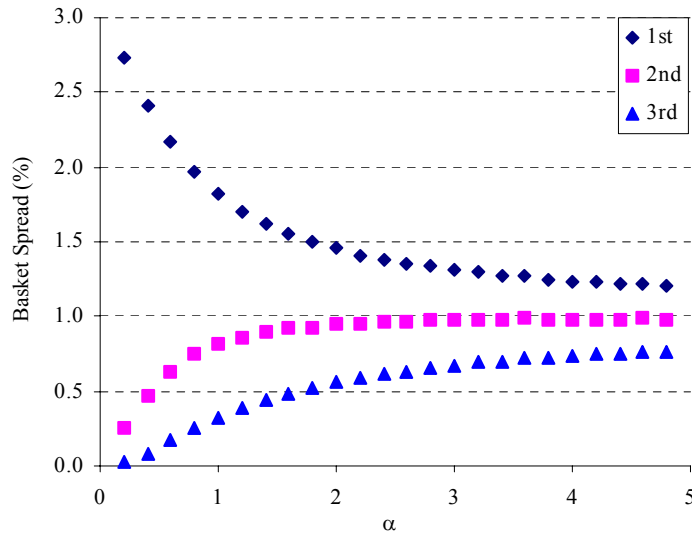
Figure 4.3: Basket spread of the reference case as function of α if joint default is modelled as Clayton copula. Maturity is set to $T = 1$ and the number of Monte Carlo simulation runs is $N = 5 \cdot 10^6$. Standard deviation: For the 1st-to-default the relative errors (standard deviation divided by the basket spread) are for all values of α smaller than 2.1%, for the 2nd-to-default smaller than 1.2% and for the 3rd-to-default smaller than 5.2%.

For a specific value of α the Clayton copula produces nearly equal results as the Gaussian normal copula. This value is $\alpha = 0.27$ for the reference case basket. The basket spreads are 2.626 for the 1st-to-default, 0.338 for the 2nd-to-default and 0.038 for the 3rd-to-default.

Again, the results for this copula show that the choice of the copula has an enormous influence on the basket spread.

4.2 Valuation of a real world Basket CDS Tranche

Given the contractual characteristics of a real basket tranche, the implemented routines can be used to find a price for this product. The basket tranche to be priced consists of 31 underlyings with individual marginal default probabilities and recovery rates. The notional per underlying is $10,000,000$ Euro and the lower limit of the tranche is $20,000,000$ Euro whereas the upper limit is $40,000,000$ Euro. The fixed and float leg type of the tranche are both recovery style. As the basket tranche to be priced is a real world product, the data has been anonymized; that is, the underlying names are called Credit 1 to Credit 31. The data can be found in Appendix C. Some key information about the tranche is listed in the subsequent Table 4.1. Using Program 2 for basket CDS tranche pricing with the data above, a present value of $1,520,203$ is computed. The difference to the price provided by the protection buyer

Property	Value
Start	02-May-2003
Maturity	02-May-2008
Coupon Frequency	3 monthly
Lower Boundary	20,000,000
Upper Boundary	40,000,000
Currency	Euro
Premium-Leg-Type	Recovery
Default-Leg-Type	Recovery
Spread	120 bps
Evaluation time (=Today)	02-May-2003
Present Value	1,529,295

Table 4.1: Basic Data of the real world basket CDS tranche.

of $9,092$ corresponds to a percentile deviation less than 1%. In the next sections this real world basket is examined[3] regarding the influence recovery rates, correlations, default probabilities and the lower and upper boundary have on the tranche spread[4].

4.2.1 Tranche Spread as Function of the Recovery Rates

In this section the influence of the recovery rate on the tranche spread is investigated. Starting with the initial recovery rates given in Appendix C the recovery rates are
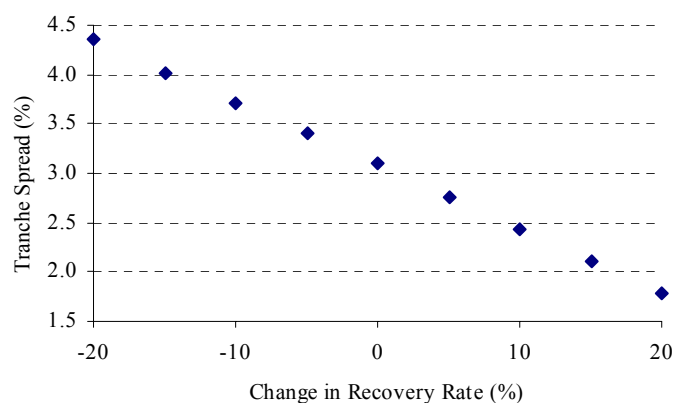
Figure 4.4: Tranche spread of the real world basket CDS tranche as function of the shift in the recovery rates. The shift is given as percentage.

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[3]Computations are done with Program 2.

[4]With the given parameters the fair tranche spread is 3.1027% for valuation date 02-May-2003.

changed[5] and the tranche spread with the changed recovery rates is computed. Figure 4.4 shows that with increasing recovery rates the tranche spread decreases. This is exactly what one would expect as with increasing recovery rates the severity of a default decreases and thus also the present value of the default leg decreases.

4.2.2 Tranche Spread as Function of the Correlations

A similar analysis as with recovery rates can be done to determine the influence the correlation has on the tranche spread. Commencing with the correlation coefficients given in Appendix C every coefficient is shifted by the same amount. Figure 4.5 shows how the spread changes if the correlations between the underlying credits are changed. For the regarded tranche an increase in correlation leads to an increase in the tranche spread.
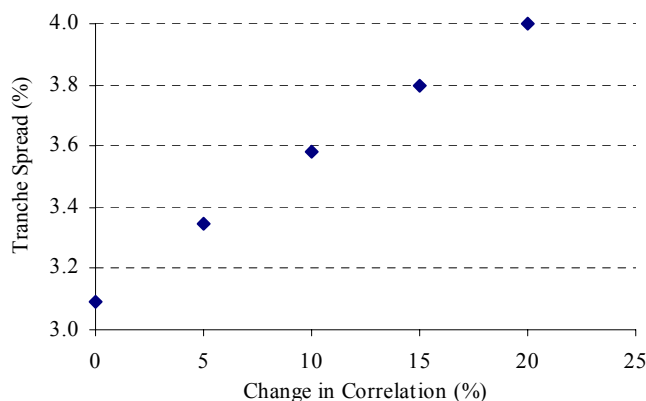
Figure 4.5: Tranche spread of the real world basket CDS tranche as function of the shift in the correlations. The shift is given as percentage.

4.2.3 Tranche Spread as Function of the Default Probabilities

The influence of the default probabilities can be investigated by changing the default probabilities for all underlyings by the same amount, calculating the new spread and comparing this changed spread with the original. Figure 4.6 shows how the spread

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[5]The recovery rates are changed on a global scale, that is, the recovery rate of all credits is shifted by the same amount. A value of 5 in figure 4.4 means that the recovery rate of all credits of the basket is increased by 5%. For example, credit 1 having an original recovery rate of 45% has a changed recovery rate of 50%.

changes if the default probabilities of all underlyings for all maturities[6] are increased by the same amount.
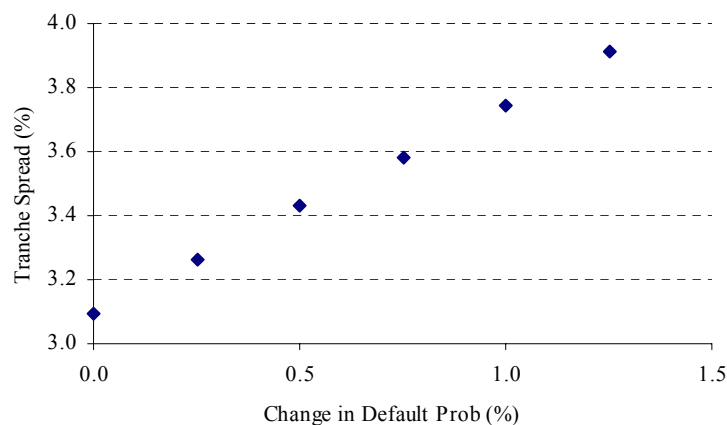
Figure 4.6: Tranche spread of the real world basket CDS tranche as function of the shift in the default probabilities. The shift is given as percentage.

As expected, an increase in the default probabilities causes an increase in the tranche spread.

4.2.4 Tranche Spread as Function of the regarded Tranche

In this section it is investigated how the tranche spread depends on the lower and upper boundary of the basket CDS tranche. Figure 4.7 shows the spread as function

Boundary		
lower	upper	Tranche Spread
0	20	10.1251
20	40	3.1027
40	60	1.4895
60	80	0.8012
80	100	0.4340
100	120	0.2132

Table 4.2: This table shows the tranche spread for different tranches.

of the lower boundary. With increasing lower boundary the tranche spread decreases. This result is quite intuitive as with an increasing lower boundary the probability that the tranche is affected by defaults decreases.

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[6]This corresponds to a parallel shift of the complete default curve.
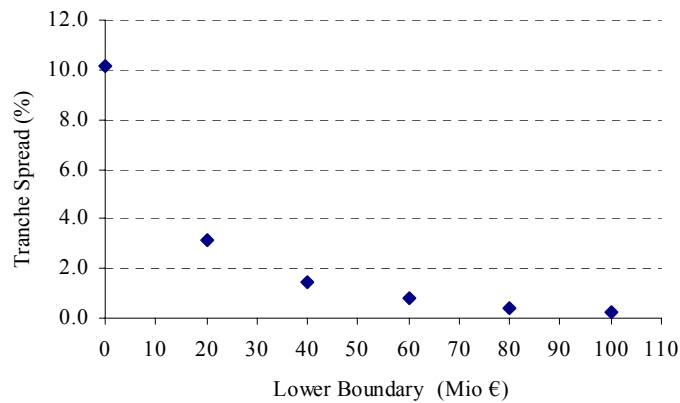
Figure 4.7: Tranche spread of the real world basket CDS tranche as function of the shift in the lower boundary of the tranche. Upper boundary is always lower boundary plus 20 Mio. Euro.

4.2.5 Student t Copula - Tranche Spread as Function of ν

In this section it is investigated what influence the chosen correlation structure has on the tranche spread[7]. In the previous sections correlated defaults are modelled via the Gaussian normal copula whereas in this section the tranche spread is determined for a Student t copula correlation structure. Figure 4.8 shows the tranche spread computed
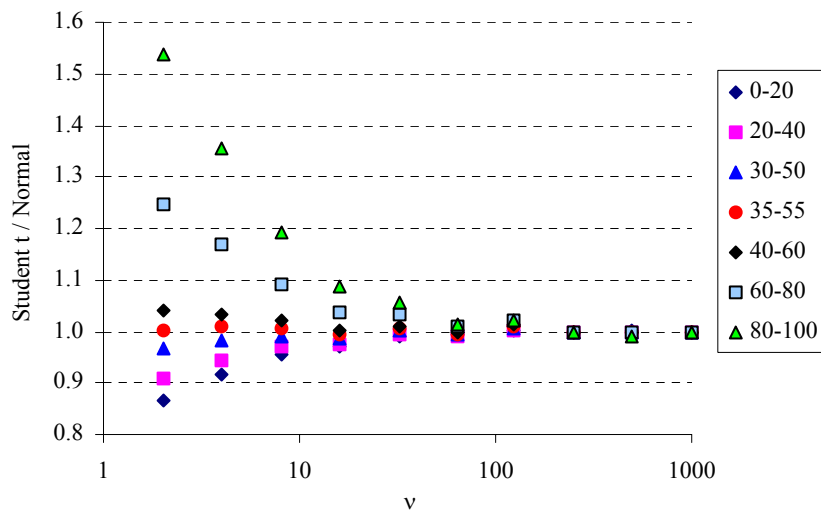
Figure 4.8: Student t copula tranche spread relative to Gaussian normal copula tranche spread as function of ν.

if correlated default is modelled via a Student t copula divided by the tranche spread computed if correlated default is modelled via Gaussian normal copula for different

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[7]Computations are done with Program 4.

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tranches. As can be seen in Figure 4.8, there are two different regimes with respect to the influence increasing ν has on the tranche spread. If the lower boundary is smaller than $35 \cdot 10^6$, then increasing ν leads to an increase in the tranche spread. This behaviour is similar to a 1st-to-default. If the lower boundary is bigger than $35 \cdot 10^6$, then increasing ν results in decreasing tranche spreads. This behaviour can be compared to 2nd-to-default-basket.

4.2.6 Clayton Copula - Tranche Spread as Function of α

In this section the influence of the parameter α of the Clayton copula on the tranche spread is examined. As already mentioned the parameter α describes dependence within the Clayton copula. Using Program 8 provides the results presented in Figure 4.9. Depending on the location of the tranche within the portfolio the influence of increasing α on the tranche spread is different. For the tranche from 0 to $20 \cdot 10^6$ Euro the tranche spread decreases with increasing α, whereas for the tranches with a lower boundary bigger than $20 \cdot 10^6$ Euro the tranche spread increases with increasing α. As
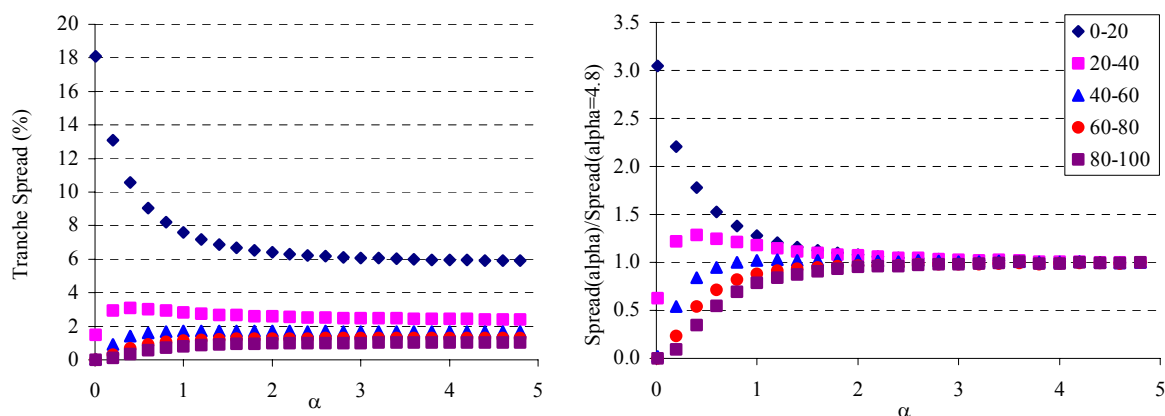
Figure 4.9: Left side: Clayton copula tranche spread as function of the parameter α. Right side: Clayton copula tranche spread at given α divided by the tranche spread at $\alpha = 4.8$.

it is difficult to see the curve shapes in the left side of the figure for higher tranches, the right side of the figure compares the curve shapes by dividing the tranche spread for a certain α with the tranche spread for $\alpha = 4.8$.

Chapter 5

Conclusion

In this thesis basket default swaps and basket CDS tranches (CDOs) are priced with two different methods. The first method is standard Monte Carlo, the second is a factor copula based approach providing semi-explicit formulae for pricing. The semi-explicit approach is implemented for a Gaussian normal copula correlation structure whereas the Monte Carlo method is implemented for a Gaussian normal-, a Student t- and a Clayton copula.

If the correlation structure of the regarded BDS is modelled with a Student t copula, higher probability mass compared to a Gaussian normal copula is found at the margins of the distribution leading to lower 1st-to-default and higher 2nd- and 3rd-to-default basket spreads. This effect is found to be especially large for the 3rd-to-default basket, where the basket spread for $\nu = 2$ is nearly 5 times the size of the basket spread if the correlation structure is modelled via a Gaussian normal copula. A qualitatively similar effect can be found in basket CDS tranches. Tranches covering the first losses also show lower tranche spreads if joint default is modelled with a Student t copula. The explanation for this behaviour is that for a Student t copula the probability of joint defaults is higher than for a Gaussian normal copula.

In order to use the semi-explicit approach, the correlation matrix has to be mapped to the correlation vector β of a factor copula model. If it is not possible to map the correlation matrix to the correlation vector for the semi-explicit approach accurately, then there is a nearly linear dependence between the error in the correlation vector determined with the least square method and the error in the calculated basket spreads.

In certain cases an accurate mapping is possible; then both methods provide equal results for basket and tranche spreads. The advantage of the semi-explicit method is that the complete loss distribution can be determined via a FFT of the characteristic

function of the loss. Compared to Monte Carlo methods this method is much faster and thus even for very large portfolios computations are possible in a reasonable time.

In this thesis a real world basket CDS tranche is priced. The determined price agrees with the price provided by the protection buyer. For the real world basket CDS tranche it is examined what influence changes in the correlation matrix, in the recovery rates, in the default probabilities and in the position of the tranche within the credit portfolio have on the tranche spread. The results agree with the expectation that

- increasing recovery rates lead to a decrease in the tranche spread

- increasing default probabilities result in an increase in the tranche spread

- a shift of the lower boundary of the tranche to higher values leads to lower tranche spreads

The influence of increasing correlation on the tranche depends on the position of the tranche within the credit portfolio. For the tranche given in the regarded real world basket CDS tranche increasing correlation leads to an increase in the basket spread.

If a Clayton copula is used to model the correlation structure, an increase in the parameter α describes an increase in the correlation. For a basket default swap 1st-to-default protection gets cheaper with increasing α whereas 2nd- and 3rd-to-default protection gets more expensive with increasing α. This observation is in good concordance with the influence the linear correlation coefficient has on the basket spread for a Gaussian normal copula. For this copula increasing correlation leads also to higher costs for 1st-to-default protection and lower costs for 2nd- and 3rd-to-default protection.

If the results gained with different copulae are compared, it is getting obvious that the choice of the used copula has an enormous influence on the determined basket or tranche spreads.

Appendix A

Derivation of the default probability $Z_k^i(t)$

In order to price an inhomogeneous BDS, the probability $Z_k^i(t)$ defined as

$$Z_k^i(t) = \lim_{t' \to t} \frac{1}{t' - t} \mathbb{P}(N_i(t') - N_i(t) = 1, N^{(-i)}(t) = k) \tag{A.1}$$

is required. $N_i(t') - N_i(t)$ gives the number of defaulted credits between t and t' and $N^{(-i)}(t) = \sum_{j \neq i} N_i(t)$ the number of defaulted credits before or at t where credit i is not taken into account. The probability $\mathbb{P}(N_i(t') - N_i(t) = 1, N^{(-i)}(t) = k)$ can be computed via the joint probability generating function of $(N_i(t') - N_i(t), N^{(-i)}(t))$:

$$\psi(u, v) = E\left[u^{N_i(t') - N_i(t)} v^{N^{(-i)}(t)} \right] \tag{A.2}$$

This equation can be written as

$$\begin{aligned}
\psi(u, v) &= \sum_{k=1}^{n} \mathbb{P}(N_i(t') - N_i(t) = 0, N^{(-i)}(t) = k) v^k \\
&+ \sum_{k=1}^{n-1} \mathbb{P}(N_i(t') - N_i(t) = 1, N^{(-i)}(t) = k) u\, v^k
\end{aligned} \tag{A.3}$$

$\psi(u, v)$ can also be stated as

$$\psi(u, v) = E\left[E\left[u^{N_i(t') - N_i(t)} v^{N^{(-i)}(t)} | V \right] \right] \tag{A.4}$$

By conditional independence this is

$$\psi(u, v) = E\left[E\left[u^{N_i(t') - N_i(t)} | V \right] \times E\left[v^{N^{(-i)}(t)} | V \right] \right] \tag{A.5}$$

With

$$\begin{aligned}
\mathbb{P}\left(N_i(t') - N_i(t) = 1 \right) &= p_{t'}^{i|V} - p_t^{i|V} \tag{A.6} \\
\mathbb{P}\left(N_i(t') - N_i(t) = 0 \right) &= 1 - \left(p_{t'}^{i|V} - p_t^{i|V} \right) \tag{A.7}
\end{aligned}$$

this can be written in terms of the conditional default probabilities $p_t^{i|V}$

$$\psi(u,v) = E\left[\left(1 - p_{t'}^{i|V} + p_t^{i|V} + \left(p_{t'}^{i|V} - p_t^{i|V}\right)u\right) \times \prod_{j\neq i}\left(1 - p_t^{j|V} + p_t^{j|V}v\right)\right] \quad (A.8)$$

Comparing equation (A.3) with (A.8), the subsequent identity is found

$$\sum_{k=1}^{n-1}\mathbb{P}(N_i(t') - N_i(t) = 1, N(t) = k)v^k = E\left[(p_{t'}^{i|V} - p_t^{i|V})\prod_{j\neq i}(1 - p_t^{j|V} + p_t^{j|V}v)\right] \quad (A.9)$$

For smooth conditional default probabilities $p_t^{i|V}$ the limit $t' \to t$ is defined for both sides:

$$\sum_{k=1}^{n-1}Z_k^i(t)v^k = E\left[\frac{dp_t^{i|V}}{dt}\prod_{j\neq i}\left(1 - p_t^{j|V} + p_t^{j|V}v\right)\right] \quad (A.10)$$

In order to determine $Z_k^i(t)$ the product on the right side of (A.10) has to be expanded. Then the expectation of the v^k term with respect to V has to be computed.

Appendix B

Limitation of Factor Models

The idea behind factor models is to reduce the complexity of the correlation structure. In certain cases this can be done without loss of accuracy, but in general it is not possible to map any correlation matrix to a correlation vector for a factor model. In this section such cases are studied, and it is investigated what impact the error in the correlation vector for the semi-explicit approach has on the calculated spread.

General Remarks The investigation is performed for a portfolio consisting of three credits with properties already described in the reference case (Schmidt and Ward). Starting from a correlation structure which can be reproduced perfectly in the factor model, the correlation matrix is changed so that the correlation structure cannot be reproduced by the correlation β vector any longer. The initial values are chosen as $\rho_{12} = 0.4$, $\rho_{13} = 0.5$ and $\rho_{23} = 0.8$ and the best correlation vector is determined with the least square method. That is, $\epsilon^2 = (\rho_{12} - \beta_1\beta_2)^2 + (\rho_{13} - \beta_1\beta_3)^2 + (\rho_{23} - \beta_2\beta_3)^2$ is minimized under the additional restriction $\beta_3 \leq 0.9999$.[1] The results of this procedure can be found in Table B.1.

In order to determine the impact of an error in the β-factors on the calculated spread in the semi-explicit approach, the difference between the spread calculated with the correct correlation matrix with a Monte Carlo simulation and the spread calculated with erroneous β-factors is computed. This difference is called basket spread error and is defined as

$$\Delta\text{Spread} = \text{Spread}_{Monte\,Carlo} - \text{Spread}_{Semi-Explicit\,Approach} \tag{B.1}$$

This procedure is performed for the 1st-, 2nd- and 3rd-to-default basket for maturity $T = 1, ..., 5$. The data obtained in this way is displayed in Table B.2. Scenario 1

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[1] This condition is required as otherwise there are problems in computing the conditional default probability $p_t^{i|V}$.

| Scenario | Monte Carlo | | | Semi-Explicit Approach | | | Error |
	ρ_{12}	ρ_{13}	ρ_{23}	β_1	β_2	β_3	ϵ^2
1	0.4	0.5	0.8	0.500016043	0.800058688	0.9999	3.39e-9
2	0.3	0.5	0.8	0.457648610	0.775042609	0.9999	0.005416045
3	0.2	0.5	0.8	0.415130051	0.753282217	0.9999	0.022103115
4	0.1	0.5	0.8	0.372251102	0.735388483	0.9999	0.050702225
5	0.0	0.5	0.8	0.328658468	0.722068881	0.9999	0.091771662

Table B.1: This table contains the correlations used for Monte Carlo simulations and the least square minimized β-factors for the semi-explicit approach.

shows a basket spread error not equal to zero. This basket spread error is regarded as constant offset and is subtracted from all basket spread errors; the corrected basket spread error values are displayed in Table B.3. It is clear that the basket spread error decreases with increasing maturity. This effect can be seen also in Figure B.1, which shows the quotient of the basket spread error at maturity T divided by the basket spread error at maturity $T = 1$ for the 1st-to-default. A similar effect is found also
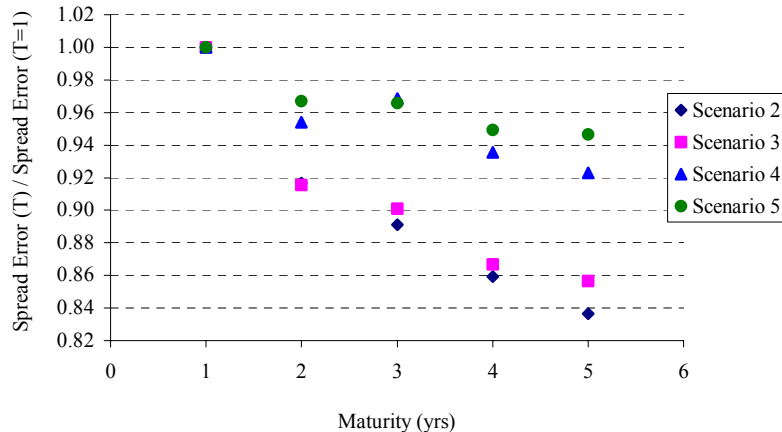
Figure B.1: This figure shows the quotient of the basket spread error at maturity T divided by the basket spread error at maturity $T = 1$.

for the 2nd- and 3rd-to-default.

It is of particular interest how the size of the β-error affects the basket spread error. The basket spread errors in Table B.3 are averaged and shown in Figure B.2 as function of the β-error. This figure implies that there is a linear dependence between the β-error and the basket spread error.

The absolute error in the basket spread is not that meaningful as one has no impression whether the very small basket spread errors of 3rd-to-default baskets related to the basket spread of the 3rd-to-default baskets are relatively big, small or
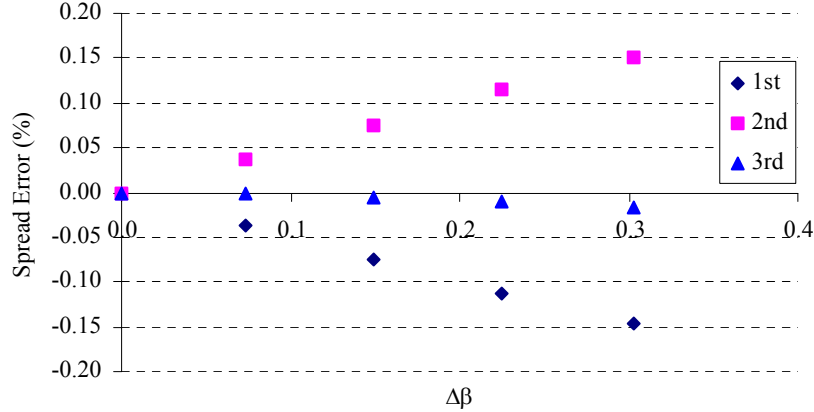
Figure B.2: Average of the basket spread errors for maturities $T = 1, ..., 5$ as function of the error in β

of comparable size. Therefore the basket spread error is translated into a property which can be understood better. The starting point is the difference between the correct basket spread s_{MC}(Monte Carlo) and the erroneous basket spread s_{SE}(Semi-explicit approach). It is possible to find a special flat correlation $\bar{\rho}_{MC}$ such that a Monte Carlo simulation with the flat correlation $\bar{\rho}_{MC}$ results in the correct basket spread s_{MC}. The same holds for the erroneous basket spread $\bar{\rho}_{SE}$: A Monte Carlo simulation with the flat correlation $\bar{\rho}_{SE}$ results in the erroneous basket spread s_{SE} : $s(MC : \rho = \bar{\rho}_{SE}) = s_{SE}$. Thus the basket spread error can be transformed into a flat correlation error: $\Delta\bar{\rho} = \bar{\rho}_{MC} - \bar{\rho}_{SE}$.

Table B.4 presents the flat correlation errors for the scenarios 1 to 5 and for the maturities $T = 1, ..., 5$. Again, scenario 1 shows values not equal to zero which are regarded as constant offset and which are subtracted from the scenarios. Table B.5 contains the corrected data. With increasing maturity the amount of the flat correlation errors decreases again. Figure B.3 displays the flat correlation errors as function of the error in β. 1st- and 2nd-to-default baskets show a linear dependence whereas the 3rd-to-default basket seems to decrease stronger than linearly as function of the β-error.
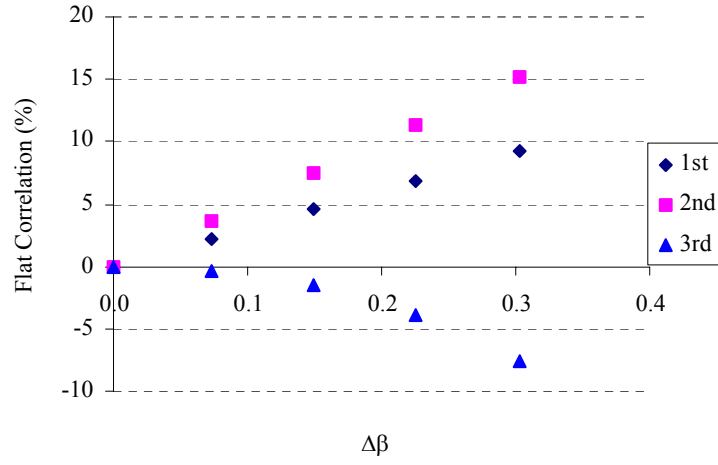
Figure B.3: Average of the errors in the flat correlation for the maturities $T = 1, ..., 5$ as function of the error in β.

Scenario	1	2	3	4	5
ϵ	0	0.0736	0.1487	0.2252	0.3029
Maturity			1st-to-default		
1	-0.0020	-0.0437	-0.0856	-0.1190	-0.1545
2	-0.0224	-0.0606	-0.0989	-0.1340	-0.1698
3	0.0067	-0.0304	-0.0686	-0.1066	-0.1405
4	0.0088	-0.0269	-0.0636	-0.1006	-0.1359
5	0.0170	-0.0179	-0.0546	-0.0909	-0.1273
			2nd-to-default		
1	0.0038	0.0438	0.0847	0.1217	0.1558
2	-0.0066	0.0319	0.0721	0.1119	0.1488
3	-0.0014	0.0346	0.0742	0.1136	0.1517
4	0.0079	0.0419	0.0788	0.1184	0.1575
5	0.0071	0.0405	0.0773	0.1157	0.1556
			3rd-to-default		
1	0.0000	-0.0010	-0.0044	-0.0077	-0.0111
2	0.0004	-0.0012	-0.0051	-0.0104	-0.0152
3	-0.0007	-0.0015	-0.0052	-0.0109	-0.0178
4	0.0016	0.0001	-0.0041	-0.0107	-0.0183
5	0.0004	-0.0013	-0.0056	-0.0121	-0.0203

Table B.2: This table shows the error in the basket spread for maturity $T = 1, ..., 5$ and order 1,2 and 3.

Scenario	1	2	3	4	5
ϵ	0	0.0736	0.1487	0.2252	0.3029
Maturity		1st-to-default			
1	0.0000	-0.0417	-0.0836	-0.1170	-0.1524
2	0.0000	-0.0382	-0.0765	-0.1116	-0.1474
3	0.0000	-0.0371	-0.0753	-0.1133	-0.1472
4	0.0000	-0.0358	-0.0724	-0.1094	-0.1447
5	0.0000	-0.0349	-0.0716	-0.1079	-0.1443
Average	0.0000	-0.0375	-0.0759	-0.1118	-0.1472
		2nd-to-default			
1	0.0000	0.0400	0.0809	0.1179	0.1521
2	0.0000	0.0385	0.0787	0.1185	0.1554
3	0.0000	0.0360	0.0756	0.1150	0.1531
4	0.0000	0.0340	0.0709	0.1105	0.1496
5	0.0000	0.0334	0.0702	0.1087	0.1485
Average	0.0000	0.0364	0.0753	0.1141	0.1517
		3rd-to-default			
1	0.0000	-0.0010	-0.0044	-0.0077	-0.0111
2	0.0000	-0.0016	-0.0055	-0.0108	-0.0156
3	0.0000	-0.0008	-0.0045	-0.0102	-0.0170
4	0.0000	-0.0015	-0.0058	-0.0123	-0.0199
5	0.0000	-0.0017	-0.0060	-0.0125	-0.0207
Average	0.0000	-0.0013	-0.0052	-0.0107	-0.0169

Table B.3: This table shows the error in the corrected basket spread: The basket spread of the first scenario (see Table B.2) is subtracted from all basket spreads.

Scenario	1	2	3	4	5
ϵ	0	0.0736	0.1487	0.2252	0.3029
Maturity	1st-to-default				
1	0.0012	0.0250	0.0519	0.0740	0.0990
2	0.0126	0.0355	0.0598	0.0840	0.1082
3	-0.0038	0.0174	0.0412	0.0656	0.0881
4	-0.0050	0.0154	0.0381	0.0615	0.0844
5	-0.0096	0.0104	0.0328	0.0556	0.0789
	2nd-to-default				
1	0.0031	0.0369	0.0721	0.1056	0.1361
2	-0.0059	0.0294	0.0668	0.1047	0.1398
3	-0.0013	0.0345	0.0742	0.1141	0.1529
4	0.0083	0.0445	0.0840	0.1265	0.1687
5	0.0079	0.0456	0.0873	0.1309	0.1762
	3rd-to-default				
1	0.0000	-0.0032	-0.0188	-0.0437	-0.0885
2	0.0008	-0.0032	-0.0166	-0.0431	-0.0793
3	-0.0014	-0.0036	-0.0148	-0.0380	-0.0756
4	0.0030	0.0003	-0.0106	-0.0332	-0.0664
5	0.0008	-0.0026	-0.0134	-0.0346	-0.0669

Table B.4: This table shows the error in the flat correlations corresponding to the basket spread for maturity $T = 1, ..., 5$ and order 1,2 and 3.

Scenario	1	2	3	4	5
ϵ	0	0.0736	0.1487	0.2252	0.3029
Maturity		1st-to-default			
1	0.0000	0.0238	0.0507	0.0728	0.0978
2	0.0000	0.0228	0.0472	0.0714	0.0955
3	0.0000	0.0212	0.0450	0.0693	0.0919
4	0.0000	0.0204	0.0431	0.0665	0.0893
5	0.0000	0.0200	0.0424	0.0651	0.0885
Average	0.0000	0.0216	0.0457	0.0690	0.0926
		2nd-to-default			
1	0.0000	0.0339	0.0691	0.1025	0.1331
2	0.0000	0.0353	0.0727	0.1107	0.1457
3	0.0000	0.0358	0.0755	0.1155	0.1542
4	0.0000	0.0362	0.0757	0.1182	0.1604
5	0.0000	0.0377	0.0794	0.1230	0.1683
Average	0.0000	0.0358	0.0745	0.1140	0.1523
		3rd-to-default			
1	0.0000	-0.0032	-0.0188	-0.0437	-0.0884
2	0.0000	-0.0041	-0.0174	-0.0440	-0.0802
3	0.0000	-0.0022	-0.0134	-0.0366	-0.0742
4	0.0000	-0.0027	-0.0136	-0.0362	-0.0694
5	0.0000	-0.0034	-0.0142	-0.0354	-0.0676
Average	0.0000	-0.0031	-0.0155	-0.0392	-0.0760

Table B.5: This table shows the error in the corrected flat correlations: The flat correlation of the first scenario (see Table B.4) is subtracted from all flat correlations.

Appendix C

Evaluation Data for the real world Basket CDS Tranche

The subsequent tables show the recovery rates, the survival probabilities of the underlying credits and the correlations between the credits as of the evaluation date 02-May-2003.

Table	Content
C.1	Recovery Rates and Survival Probabilities from 3m to 18m
C.2	Survival Probabilities from 21m to 39m
C.3	Survival Probabilities from 42m to 60m
C.4	Correlations
C.5	Correlations
C.6	Correlations

		Survival Probability					
Underlying	Recovery	3m	6m	9m	12m	15m	18m
Credit 1	0.45	0.9996	0.9992	0.9986	0.9980	0.9973	0.9966
Credit 2	0.45	0.9984	0.9968	0.9952	0.9938	0.9923	0.9909
Credit 3	0.45	0.9984	0.9968	0.9952	0.9938	0.9923	0.9909
Credit 4	0.45	0.9950	0.9897	0.9840	0.9779	0.9717	0.9654
Credit 5	0.23	0.9989	0.9978	0.9963	0.9946	0.9926	0.9905
Credit 6	0.23	0.9994	0.9987	0.9978	0.9968	0.9956	0.9943
Credit 7	0.23	0.9994	0.9988	0.9979	0.9968	0.9954	0.9938
Credit 8	0.23	0.9973	0.9944	0.9915	0.9886	0.9857	0.9827
Credit 9	0.23	0.9992	0.9983	0.9972	0.9959	0.9944	0.9927
Credit 10	0.45	0.9979	0.9956	0.9932	0.9908	0.9882	0.9855
Credit 11	0.23	0.9981	0.9960	0.9936	0.9910	0.9881	0.9851
Credit 12	0.45	0.9993	0.9986	0.9975	0.9962	0.9947	0.9931
Credit 13	0.45	0.9996	0.9993	0.9988	0.9982	0.9976	0.9969
Credit 14	0.45	0.9944	0.9884	0.9829	0.9775	0.9723	0.9673
Credit 15	0.45	0.9956	0.9909	0.9860	0.9807	0.9752	0.9695
Credit 16	0.23	0.9989	0.9977	0.9965	0.9951	0.9936	0.9920
Credit 17	0.45	0.9988	0.9975	0.9957	0.9935	0.9911	0.9885
Credit 18	0.45	0.9984	0.9968	0.9952	0.9938	0.9924	0.9910
Credit 19	0.45	0.9996	0.9993	0.9985	0.9975	0.9963	0.9948
Credit 20	0.45	0.9961	0.9919	0.9875	0.9829	0.9781	0.9733
Credit 21	0.45	0.9828	0.9648	0.9474	0.9303	0.9135	0.8970
Credit 22	0.45	0.9993	0.9985	0.9977	0.9969	0.9961	0.9952
Credit 23	0.45	0.9966	0.9931	0.9893	0.9854	0.9813	0.9772
Credit 24	0.23	0.9992	0.9985	0.9976	0.9966	0.9956	0.9945
Credit 25	0.45	0.9862	0.9717	0.9576	0.9437	0.9301	0.9166
Credit 26	0.45	0.9996	0.9992	0.9987	0.9982	0.9976	0.9970
Credit 27	0.45	0.9957	0.9912	0.9869	0.9827	0.9786	0.9746
Credit 28	0.45	0.9968	0.9933	0.9896	0.9856	0.9813	0.9769
Credit 29	0.45	0.9970	0.9938	0.9906	0.9873	0.9841	0.9809
Credit 30	0.23	0.9993	0.9986	0.9978	0.9969	0.9959	0.9949
Credit 31	0.45	0.9826	0.9646	0.9472	0.9303	0.9138	0.8978

Table C.1: Recovery Rate and survival probability for 3m to 18m.

Underlying	Survival Probability						
	21m	24m	27m	30m	33m	36m	39m
Credit 1	0.9958	0.9949	0.9939	0.9929	0.9918	0.9907	0.9897
Credit 2	0.9894	0.9881	0.9865	0.9848	0.9831	0.9815	0.9795
Credit 3	0.9894	0.9881	0.9865	0.9848	0.9831	0.9815	0.9795
Credit 4	0.9585	0.9519	0.9452	0.9381	0.9312	0.9243	0.9174
Credit 5	0.9880	0.9855	0.9828	0.9798	0.9767	0.9735	0.9701
Credit 6	0.9928	0.9912	0.9896	0.9878	0.9859	0.9840	0.9819
Credit 7	0.9920	0.9900	0.9880	0.9857	0.9833	0.9808	0.9783
Credit 8	0.9796	0.9767	0.9734	0.9700	0.9667	0.9634	0.9596
Credit 9	0.9908	0.9888	0.9868	0.9845	0.9821	0.9797	0.9770
Credit 10	0.9826	0.9797	0.9767	0.9735	0.9702	0.9670	0.9634
Credit 11	0.9818	0.9784	0.9749	0.9711	0.9672	0.9632	0.9591
Credit 12	0.9912	0.9892	0.9872	0.9849	0.9827	0.9803	0.9777
Credit 13	0.9962	0.9954	0.9946	0.9937	0.9927	0.9918	0.9907
Credit 14	0.9624	0.9579	0.9535	0.9492	0.9453	0.9416	0.9378
Credit 15	0.9632	0.9571	0.9507	0.9439	0.9371	0.9304	0.9231
Credit 16	0.9902	0.9884	0.9865	0.9844	0.9823	0.9801	0.9777
Credit 17	0.9854	0.9822	0.9791	0.9756	0.9721	0.9684	0.9647
Credit 18	0.9896	0.9884	0.9870	0.9857	0.9844	0.9831	0.9817
Credit 19	0.9931	0.9912	0.9894	0.9874	0.9852	0.9830	0.9807
Credit 20	0.9680	0.9630	0.9576	0.9518	0.9460	0.9402	0.9340
Credit 21	0.8803	0.8649	0.8493	0.8335	0.8185	0.8042	0.7892
Credit 22	0.9943	0.9934	0.9925	0.9915	0.9906	0.9896	0.9887
Credit 23	0.9727	0.9683	0.9637	0.9588	0.9539	0.9490	0.9436
Credit 24	0.9933	0.9920	0.9907	0.9893	0.9878	0.9863	0.9847
Credit 25	0.9029	0.8902	0.8773	0.8642	0.8517	0.8397	0.8272
Credit 26	0.9964	0.9957	0.9950	0.9941	0.9932	0.9923	0.9913
Credit 27	0.9706	0.9670	0.9633	0.9596	0.9562	0.9529	0.9496
Credit 28	0.9720	0.9672	0.9623	0.9570	0.9517	0.9464	0.9406
Credit 29	0.9775	0.9743	0.9709	0.9674	0.9640	0.9607	0.9572
Credit 30	0.9938	0.9926	0.9913	0.9900	0.9885	0.9871	0.9855
Credit 31	0.8817	0.8670	0.8521	0.8372	0.8230	0.8097	0.7961

Table C.2: Survival probability for 21m to 39m.

Underlying	Survival Probability						
	42m	45m	48m	51m	54m	57m	60m
Credit 1	0.9887	0.9877	0.9867	0.9855	0.9842	0.9829	0.9815
Credit 2	0.9774	0.9752	0.9731	0.9706	0.9680	0.9653	0.9626
Credit 3	0.9774	0.9752	0.9731	0.9706	0.9680	0.9653	0.9626
Credit 4	0.9107	0.9040	0.8973	0.8906	0.8839	0.8772	0.8705
Credit 5	0.9667	0.9631	0.9594	0.9557	0.9519	0.9480	0.9440
Credit 6	0.9798	0.9776	0.9753	0.9731	0.9707	0.9683	0.9658
Credit 7	0.9757	0.9729	0.9701	0.9672	0.9643	0.9612	0.9579
Credit 8	0.9559	0.9520	0.9481	0.9442	0.9402	0.9361	0.9319
Credit 9	0.9744	0.9715	0.9686	0.9658	0.9629	0.9599	0.9568
Credit 10	0.9598	0.9561	0.9523	0.9486	0.9448	0.9409	0.9370
Credit 11	0.9550	0.9507	0.9464	0.9421	0.9377	0.9332	0.9286
Credit 12	0.9750	0.9721	0.9692	0.9665	0.9638	0.9609	0.9580
Credit 13	0.9897	0.9886	0.9875	0.9865	0.9855	0.9844	0.9834
Credit 14	0.9343	0.9309	0.9278	0.9251	0.9224	0.9200	0.9179
Credit 15	0.9158	0.9083	0.9008	0.8932	0.8854	0.8775	0.8694
Credit 16	0.9753	0.9728	0.9702	0.9676	0.9647	0.9618	0.9588
Credit 17	0.9609	0.9569	0.9528	0.9489	0.9449	0.9408	0.9366
Credit 18	0.9803	0.9789	0.9775	0.9759	0.9743	0.9726	0.9710
Credit 19	0.9784	0.9760	0.9735	0.9710	0.9685	0.9659	0.9631
Credit 20	0.9278	0.9214	0.9150	0.9082	0.9012	0.8940	0.8867
Credit 21	0.7750	0.7608	0.7472	0.7337	0.7204	0.7072	0.6945
Credit 22	0.9878	0.9870	0.9861	0.9851	0.9840	0.9828	0.9817
Credit 23	0.9381	0.9324	0.9266	0.9208	0.9149	0.9088	0.9026
Credit 24	0.9830	0.9813	0.9796	0.9777	0.9758	0.9738	0.9717
Credit 25	0.8152	0.8032	0.7917	0.7803	0.7688	0.7576	0.7466
Credit 26	0.9903	0.9893	0.9882	0.9870	0.9858	0.9846	0.9833
Credit 27	0.9465	0.9434	0.9405	0.9377	0.9350	0.9323	0.9299
Credit 28	0.9350	0.9291	0.9231	0.9173	0.9114	0.9054	0.8993
Credit 29	0.9538	0.9504	0.9470	0.9436	0.9402	0.9367	0.9332
Credit 30	0.9839	0.9823	0.9806	0.9789	0.9772	0.9755	0.9737
Credit 31	0.7832	0.7706	0.7586	0.7466	0.7348	0.7233	0.7122

Table C.3: Survival probability for 42m to 60m.

		Credit								
	1	2	3	4	5	6	7	8	9	10
Credit 1	1	0.651	0.607	0.225	0.751	0.710	0.595	0.647	0.641	0.432
Credit 2	0	1.000	0.848	0.309	0.796	0.808	0.701	0.686	0.529	0.575
Credit 3	0	0	1.000	0.319	0.702	0.680	0.615	0.611	0.401	0.498
Credit 4	0	0	0	1.000	0.301	0.274	0.301	0.269	0.233	0.332
Credit 5	0	0	0	0	1.000	0.909	0.808	0.878	0.848	0.661
Credit 6	0	0	0	0	0	1.000	0.730	0.714	0.693	0.582
Credit 7	0	0	0	0	0	0	1.000	0.690	0.699	0.596
Credit 8	0	0	0	0	0	0	0	1.000	0.667	0.644
Credit 9	0	0	0	0	0	0	0	0	1.000	0.498
Credit 10	0	0	0	0	0	0	0	0	0	1.000
Credit 11	0	0	0	0	0	0	0	0	0	0
Credit 12	0	0	0	0	0	0	0	0	0	0
Credit 13	0	0	0	0	0	0	0	0	0	0
Credit 14	0	0	0	0	0	0	0	0	0	0
Credit 15	0	0	0	0	0	0	0	0	0	0
Credit 16	0	0	0	0	0	0	0	0	0	0
Credit 17	0	0	0	0	0	0	0	0	0	0
Credit 18	0	0	0	0	0	0	0	0	0	0
Credit 19	0	0	0	0	0	0	0	0	0	0
Credit 20	0	0	0	0	0	0	0	0	0	0
Credit 21	0	0	0	0	0	0	0	0	0	0
Credit 22	0	0	0	0	0	0	0	0	0	0
Credit 23	0	0	0	0	0	0	0	0	0	0
Credit 24	0	0	0	0	0	0	0	0	0	0
Credit 25	0	0	0	0	0	0	0	0	0	0
Credit 26	0	0	0	0	0	0	0	0	0	0
Credit 27	0	0	0	0	0	0	0	0	0	0
Credit 28	0	0	0	0	0	0	0	0	0	0
Credit 29	0	0	0	0	0	0	0	0	0	0
Credit 30	0	0	0	0	0	0	0	0	0	0
Credit 31	0	0	0	0	0	0	0	0	0	0

Table C.4: Table of correlations.

	Credit									
	11	12	13	14	15	16	17	18	19	20
1	0.582	0.650	0.560	0.397	0.660	0.584	0.576	0.586	0.654	0.444
2	0.692	0.542	0.543	0.544	0.747	0.627	0.672	0.739	0.769	0.506
3	0.640	0.461	0.566	0.604	0.612	0.525	0.599	0.780	0.634	0.389
4	0.221	0.364	0.405	0.380	0.281	0.350	0.241	0.258	0.278	0.235
5	0.776	0.706	0.521	0.463	0.854	0.673	0.778	0.815	0.822	0.569
6	0.747	0.643	0.468	0.480	0.835	0.686	0.791	0.695	0.911	0.610
7	0.683	0.611	0.472	0.490	0.700	0.505	0.676	0.666	0.672	0.441
8	0.623	0.593	0.460	0.430	0.714	0.597	0.618	0.681	0.641	0.411
9	0.648	0.700	0.390	0.253	0.702	0.511	0.640	0.518	0.621	0.484
10	0.484	0.443	0.424	0.426	0.523	0.565	0.508	0.570	0.523	0.369
11	1.000	0.591	0.350	0.381	0.730	0.533	0.726	0.650	0.664	0.516
12	0	1.000	0.422	0.324	0.680	0.452	0.614	0.473	0.572	0.456
13	0	0	1.000	0.632	0.398	0.529	0.439	0.493	0.436	0.348
14	0	0	0	1.000	0.380	0.484	0.458	0.445	0.500	0.188
15	0	0	0	0	1.000	0.620	0.788	0.685	0.750	0.640
16	0	0	0	0	0	1.000	0.614	0.514	0.659	0.524
17	0	0	0	0	0	0	1.000	0.617	0.740	0.584
18	0	0	0	0	0	0	0	1.000	0.633	0.432
19	0	0	0	0	0	0	0	0	1.000	0.631
20	0	0	0	0	0	0	0	0	0	1.000
21	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0

Table C.5: Table of correlations.

	Credit										
	21	22	23	24	25	26	27	28	29	30	31
1	0.411	0.580	0.571	0.403	0.660	0.506	0.129	0.602	0.539	0.509	0.614
2	0.333	0.492	0.750	0.503	0.832	0.488	0.175	0.714	0.546	0.697	0.739
3	0.159	0.439	0.652	0.461	0.723	0.410	0.228	0.608	0.442	0.569	0.659
4	0.153	0.239	0.265	0.386	0.353	0.123	0.285	0.189	0.254	0.218	0.316
5	0.375	0.615	0.642	0.461	0.859	0.576	0.113	0.786	0.562	0.616	0.776
6	0.376	0.582	0.657	0.465	0.864	0.613	0.117	0.792	0.640	0.678	0.760
7	0.302	0.461	0.561	0.433	0.805	0.457	0.151	0.648	0.455	0.465	0.714
8	0.357	0.488	0.552	0.367	0.724	0.421	0.146	0.679	0.408	0.494	0.684
9	0.372	0.585	0.452	0.385	0.662	0.501	0.084	0.582	0.472	0.428	0.585
10	0.347	0.389	0.542	0.375	0.621	0.328	0.106	0.461	0.335	0.405	0.626
11	0.228	0.504	0.505	0.367	0.714	0.552	0.045	0.700	0.494	0.554	0.685
12	0.438	0.493	0.458	0.412	0.625	0.461	0.088	0.564	0.472	0.383	0.562
13	0.324	0.421	0.554	0.509	0.556	0.321	0.259	0.355	0.416	0.444	0.492
14	0.261	0.263	0.588	0.678	0.528	0.202	0.381	0.312	0.415	0.449	0.547
15	0.336	0.592	0.596	0.470	0.802	0.572	0.066	0.937	0.560	0.652	0.721
16	0.220	0.599	0.570	0.430	0.706	0.482	0.201	0.592	0.498	0.614	0.568
17	0.203	0.458	0.599	0.493	0.765	0.593	0.104	0.783	0.603	0.649	0.711
18	0.146	0.447	0.551	0.357	0.704	0.425	0.027	0.658	0.372	0.508	0.646
19	0.341	0.546	0.622	0.445	0.797	0.593	0.086	0.710	0.676	0.705	0.708
20	0.227	0.626	0.380	0.277	0.581	0.529	-0.019	0.619	0.733	0.653	0.510
21	1.000	0.269	0.450	0.326	0.307	0.103	0.113	0.249	0.430	0.243	0.326
22	0	1.000	0.387	0.276	0.540	0.513	0.076	0.528	0.516	0.514	0.428
23	0	0	1.000	0.617	0.677	0.438	0.277	0.540	0.523	0.580	0.709
24	0	0	0	1.000	0.484	0.233	0.333	0.375	0.368	0.407	0.510
25	0	0	0	0	1.000	0.566	0.153	0.763	0.568	0.674	0.795
26	0	0	0	0	0	1.000	0.006	0.551	0.534	0.528	0.461
27	0	0	0	0	0	0	1.000	0.047	0.120	0.146	0.177
28	0	0	0	0	0	0	0	1.000	0.504	0.639	0.665
29	0	0	0	0	0	0	0	0	1.000	0.724	0.565
30	0	0	0	0	0	0	0	0	0	1.000	0.605
31	0	0	0	0	0	0	0	0	0	0	1.000

Table C.6: Table of correlations.

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