HANDBOOK OF MODERN FINANCE

THIRD EDITION

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A4

Mathematics of Finance: Money and Time

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Month	*Beginning Balance*	*Interest Payment*	*Principal Repayment*	*Ending Balance*
1	$1,200.00	$ 17.97	$ 92.03	$1,107.97
2	1,107.97	16.59	93.41	1,014.56
3	1,014.56	15.19	94.81	919.75
4	919.75	13.77	96.23	823.52
5	823.52	12.33	97.67	725.85
6	725.85	10.87	99.13	626.72
7	626.72	9.39	100.61	526.11
8	526.11	7.88	102.12	423.99
9	423.99	6.35	103.65	320.24
10	320.34	4.80	105.20	215.14
11	215.14	3.22	106.78	108.36
12	108.36	1.64	108.36	0.00
		$120.00	$1,200.00	

A4.03 BOND ANALYSIS: NON-INTEREST-BEARING SECURITIES

Non-interest-bearing securities are also referred to as discounted securities. Unlike regular corporate bonds, which pay periodic interest (i.e., pay a coupon), interest is earned on these bonds by their appreciation in price over time. That is, these securities originally sell for less than their maturity or face value. All other factors being constant, the price approaches the face value as the time to maturity approaches zero. Note that this is not the same thing as a coupon bond that is selling at a discount.

[1] Definitions of Variables

d = discount rate as a percentage of the face value

D = dollar value of the discount

F = face or maturity value of the security

P = current price of the security

Y = equivalent bond yield of the security

i_J = simple interest rate using a J-day year

P' = price per dollar of face value

t_M = time remaining until maturity

t_H = time held by the original investor

[2] Discount Rates and Pricing

The discount rate determines the dollar value of the discount or the difference between the price and the maturity value. At times, this is also referred to as the quoted yield of the security. For most securities, such as Treasury bills and commercial paper, the discount rate is stated as a 360-day-year annual rate. All subsequent calculations must take into account the actual number of days until maturity.

[a] Calculation of the Dollar Discount. As stated previously, the discount rate is based on a 360-day year. The dollar discount is determined by the discount rate, the face value of the security, and the number of days until maturity. The general equation for the dollar discount is

$$D = F(d)\left(\frac{t_M}{360}\right)$$

EXAMPLE: An investor has just purchased a 6-month (180-day) Treasury bill. The face value of this security is $10,000, and the discount rate is 9.75 percent. What is the dollar value of the discount for this security?

$$D = \$10{,}000(0.0975)\left(\frac{180}{360}\right) = \$487.50$$

Intuitively, this security promises to pay a discount of 9.75 percent or $975 if held for one year. Since the investor will hold this security for half a year, it is entitled to half of the discount, or $487.50.

[b] Calculation of Price. The price of a discounted security is equal to the face value minus the dollar discount. It may be calculated by either of the following two formulas:

$$P = F - D$$

$$P = F\left[1 - \frac{d(t_M)}{360}\right]$$

EXAMPLE: What is the price of a 180-day, $10,000 face value Treasury bill if the discount rate is 9.75 percent?

METHOD 1

The dollar discount associated with this security, calculated in the previous section, is $487.50. Therefore, the price is equal to

$$P = \$10{,}000 - \$487.50 = \$9{,}512.50$$

METHOD 2

The price may also be calculated as follows:

$$P = \$10{,}000\left[1 - \frac{0.0975(180)}{360}\right] = \$9{,}512.50$$

[c] Calculation of the Discount Rate. Given the face value of the security, the time to maturity, and either the dollar discount or the price, the annual discount rate may be calculated using the following equations:

$$d = \frac{D}{F}\left(\frac{360}{t_M}\right)$$

$$d = \left(\frac{F - P}{F}\right)\left(\frac{360}{t_M}\right)$$

EXAMPLE: What is the annual discount rate for a 180-day Treasury bill with a face value of \$10,000 and a current price of \$9,512.50?

METHOD 1

The dollar discount is equal to

$$D = \$10{,}000 - \$9{,}512.50 = \$487.50$$

This implies that the discount rate is equal to

$$d = \left(\frac{\$487.50}{\$10{,}000}\right)\left(\frac{360}{180}\right) = 0.0975 = 9.75\%$$

METHOD 2

The discount rate may also be calculated as

$$d = \left(\frac{\$10{,}000 - \$9{,}512.50}{\$10{,}000}\right)\left(\frac{360}{180}\right) = 0.0975 = 9.75\%$$

[3] Equivalent Simple Interest Rate

Since an investor pays less than the face value of the discounted security, the interest rate earned, which is based on price, must be greater than the discount rate, which is based on the face value. The usual procedure is to state this simple interest rate on the basis of a 365-day year.

[a] Calculations. The simple interest rate is a function of the original price paid and the amount of price appreciation earned over the investment horizon. Allowing the subscripts b and s to represent the *buying price* and the *selling price*, the simple interest rate may be calculated as

$$i_{365} = \left(\frac{P_s - P_b}{P_b}\right)\left(\frac{365}{t_H}\right)$$

If an investor holds the security until it matures, the price appreciation will be equal to the dollar discount at time of purchase; no return is earned from changes in the general level of market interest rates. The simple interest rate may then be calculated as

$$i_{365} = \frac{365(d)}{[360 - d(t_M)]}$$

EXAMPLE: If an investor purchases a 90-day Treasury bill for \$9,500 and sells it in 45 days for \$9,750, what is the simple interest rate, or the rate of return, that the investor earns on this investment?

$$i_{365} = \left(\frac{\$9{,}750 - \$9{,}500}{\$9{,}500}\right)\left(\frac{365}{45}\right) = 0.21345 = 21.345\%$$

EXAMPLE: An investor purchases a 180-day Treasury bill at a price of $9,512.50. This Treasury bill has a face value of $10,000 and a discount rate of 9.75 percent. If the investor holds this security until it matures, what is the simple interest rate on this investment?

METHOD 1

The simple interest rate may be calculated as

$$i_{365} = \left(\frac{\$10{,}000 - \$9{,}512.50}{\$9{,}512.50}\right)\left(\frac{365}{180}\right) = 0.1039 = 10.39\%$$

METHOD 2

The simple interest rate may also be calculated as follows:

$$i_{365} = \frac{365(0.0975)}{360 - 0.0975(180)} = 0.1039 = 10.39\%$$

[4] Effective Return

The simple interest rate that was calculated in the previous section assumed an add-on type of interest. The distinction between add-on interest and compound interest is not really important except in the direct comparison of securities with different maturities. In other words, an investor must be concerned with the actual investment horizon.

[a] Calculations. Assume that an investor has an investment horizon of 120 days, and that 120-day Treasury bills have a discount rate of 10 percent and 30-day Treasury bills have a discount rate of 10.25 percent. The equivalent simple interest rate for these two securities would be

$$\text{120-day: } i_{365} = \frac{365(0.1)}{360 - 0.1(120)} = 0.1049 = 10.49\%$$

$$\text{30-day: } i_{365} = \frac{365(0.1025)}{360 - 0.1025(30)} = 0.1048 = 10.48\%$$

For the 120-day Treasury bill, the simple interest rate of 10.49 percent assumes that the investor could earn

$$0.1049\left(\frac{120}{365}\right) = 0.0345 = 3.45\%$$

every 120 days. For the 30-day security, the simple interest rate of 10.48 percent assumes that the investor could earn

$$0.1048\left(\frac{30}{365}\right) = 0.0086 = 0.86\%$$

every 30 days.

From the simple interest rates of 10.49 percent and 10.48 percent, it might appear that the investor would be better off purchasing the 120-day security. However, assume that over the 120-day investment horizon the investor believes that it could purchase either the 120-day security at a discount rate of 10 percent or 4 consecutive 30-day securities each at a discount rate of 10.25 percent. In the second case, 1 security matures every 30 days, and the proceeds are reinvested in a new 30-day security, so the investor actually earns compound interest. The effective rate of return for the 30-day Treasury bills is equal to

$$\text{30-day: } i_{365} = \left\{\left[1 + \left(\frac{0.1048}{365}\right)(30)\right]^{120/30} - 1\right\}\left(\frac{365}{120}\right) = 0.1062 = 10.62\%$$

Because of compounding, the investor is better off, in terms of effective return, if it is able to invest in 4 consecutive 30-day Treasury bills, each with a discount rate of 10.25 percent.

If the investment horizon is 360 days and if each of these securities could be rolled over at their stated discount rate, the effective rates of return would be

$$\text{120-day: } i_{365} = \left\{\left[1 + \left(\frac{0.1049}{365}\right)(120)\right]^{360/120} - 1\right\}\left(\frac{365}{360}\right) = 0.1086 = 10.86\%$$

$$\text{30-day: } i_{365} = \left\{\left[1 + \left(\frac{0.1048}{365}\right)(30)\right]^{360/30} - 1\right\}\left(\frac{365}{360}\right) = 0.1099 = 10.99\%$$

[5] Equivalent Bond Yield

The equivalent bond yield, also referred to as the coupon yield equivalent, is reported on most bond dealers' quote sheets. Its purpose is to make possible direct comparisons between the interest earned on discount securities and the yield to maturity on coupon-paying bonds. Since most coupon securities pay interest on a semiannual basis, the time to maturity for the discount security is important in determining its equivalent bond yield.

[a] Calculations: Fewer Than 182 Days. A coupon bond with fewer than 182 days to maturity will make no coupon payments until it matures. Thus, its yield to maturity is already on an equivalent basis to the simple interest rate of a discounted security, and the discounted security's equivalent bond yield is equal to its simple interest rate.

EXAMPLE: What is the equivalent bond yield for a 95-day Treasury bill with a quoted discount rate of 11.5 percent?

Since the time to maturity for this security is less than 182 days, the equivalent bond yield is equal to the simple interest rate, which is equal to

$$Y = \frac{365(0.115)}{360 - (0.115)(95)} = 0.1202 = 12.02\%$$

[b] Calculations: More Than 182 Days. A coupon bond with more than 182 days to maturity will make a coupon payment before it matures. To calculate the equivalent bond yield for a discounted security with more than 182 days to mature, it must be treated as if it too paid interest on a semiannual basis. The formula for the equivalent bond yield is

$$Y = \frac{-[2(t_M)/365] + 2((t_M/365)^2 - \{[2(t_M)/365] - 1\}[1 - (1/P')])^{1/2}}{\{[2(t_M)/365] - 1\}}$$

Example: A 225-day Treasury bill has a quoted discount rate of 12 percent. What is the equivalent bond yield on this security?

Step 1. $P' = \left[1 - \frac{0.12(225)}{360}\right] = 0.925$

Step 2. The equivalent bond yield is then equal to

$$Y = \frac{-[2(225)/365] + 2((225/365)^2 - \{[2(225)/365] - 1\}[1 - (1/0.925)])^{1/2}}{\{[2(225)/365] - 1\}}$$

$= 0.1299 = 12.99\%$

This says that there would be no difference, all other factors being held constant, between holding a 225-day Treasury bill with a discount rate of 12 percent and holding a 225-day coupon bond with a yield to maturity of 12.99 percent.

A4.04 BOND ANALYSIS: INTEREST-BEARING SECURITIES

Most corporate bonds, as well as municipals and Treasury notes and bonds, pay interest on a semiannual basis. To find the interest paid during the year, multiply the par value (face or maturity value) of the bond by the annual coupon rate. This amount is then divided by 2 to determine the amount of interest paid every 6 months. It is important to know that corporate securities use a 180-day coupon period—a commercial year of 360 days or 30 days per month, whereas government securities use an exact year of 365 days (366 days for a leap year). In addition, corporate securities are delivered 5 business days after the sale, whereas government securities are delivered the same day or the day after the sale. Prices for these securities are calculated as of the delivery date. Finally, if a bond is sold between coupon dates, it will have accrued interest since the last coupon date. This accrued interest must be added to the quoted price to determine the actual amount that the investor is required to pay.

[1] Definitions of Variables

C = annual coupon rate

F = face, par, or maturity value of the bond

P = quoted price for the bond

P' = price per dollar of face value

N = number of coupon periods remaining

Y = yield to maturity or the discount rate used in present value calculations

AI = amount of accured interest

AI′ = accrued interest per dollar of face value

AY = approximation of the actual yield

CY = approximation of the yield to call

RY = approximation of the actual realized yield
RCY = realized compounded yield
SF = sinking fund payment
TVR = terminal value ratio of cash flows at maturity to original price
t_{PM} = time from purchase until maturity in days
t_{PS} = time from purchase until settlement of maturity in days
t_H = time held by the original investor since the last coupon date
t_C = time remaining until call in years
t_M = time remaining until maturity in years
t_{IH} = investor's time horizon
t_{CD} = actual number of days between coupon periods
PV = present value of future cash flows
PVIF($Y/2\%$, N) = present value interest factor for a single sum at an annual yield of Y percent for N six-month periods
PVIF$_a$($Y/2\%$, N) = present value interest factor for an annuity at an annual yield of Y percent for N six-month periods

[2] Accrued Interest

When a bond is sold between coupon dates, the price actually paid by an investor is equal to the present value of all future cash flows to be received. This value is greater than the quoted price by the amount of accrued interest, where the accrued interest is equal to the portion of the next coupon to be received that is owed to the original owner of the bond. The actual calculation of accrued interest depends on whether the security is a corporate security or a government security.

[a] Calculations: Corporate Securities. Accrued interest on corporate securities is calculated using a 180-day coupon period. All months within the coupon period are assumed to have 30 days. The accrued interest may be calculated as follows:

$$AI = F\left(\frac{C}{2}\right)\left(\frac{t_H}{180}\right)$$

Example: Assume that a 9 percent coupon bond with a face value of \$1,000 has just been purchased. Interest on this bond is paid semiannually, and it has been 45 days since the last coupon payment. What is the accrued interest on this bond?

$$AI = \$1{,}000\left(\frac{0.09}{2}\right)\left(\frac{45}{180}\right) = \$11.25$$

This amount is added to the bond's quoted price to determine the amount actually paid by the purchaser.

[b] Calculations: Government Securities. Unlike corporate securities, government securities use the actual number of days within the coupon period to determine the amount of accrued interest. This period may range from 181 days to 184 days. The accrued interest may be calculated as follows:

$$\text{AI} = F\left(\frac{C}{2}\right)\left(\frac{t_H}{t_{\text{CD}}}\right)$$

EXAMPLE: A Treasury bond with a face value of $100,000 is issued with a coupon rate of 8.75 percent. Coupon payment dates for this bond are November 15 and May 15. If this bond is purchased on January 5, what is the value of accrued interest?

There are 181 days between November 15 and May 15, and 184 days between May 15 and November 15. If the date of sale is included, there are 51 days between November 15 and January 5. The amount of accrued interest may be calculated as follows:

$$\text{AI} = \$100{,}000\left(\frac{0.0875}{2}\right)\left(\frac{51}{181}\right) = \$1{,}232.73$$

[3] Yield to Maturity

The yield to maturity is the discount rate that is used to determine the present value of all future cash flows to be received. The yield is reported on an annual basis but is an add-on interest rate. That is, one half of the reported yield is the correct rate to use per six-month period for coupon bonds with semiannual payments of interest.

[a] Calculation of Yield. For interest-bearing securities, the calculation of yield to maturity is the same as finding the internal rate of return (discount rate) for a complex cash flow. For instance, assume that a five-year, 9 percent coupon, corporate bond is purchased for $961.39. If this bond pays interest on a semiannual basis, what is its annual yield?

METHOD 1

A 9 percent coupon rate implies that this bond will pay

$$\$1{,}000\left(\frac{0.09}{2}\right) = \$45$$

every 6 months for 5 years (10 6-month periods). In addition, since it is a corporate bond, it will pay a maturity value of $1,000. The pricing equation for this bond is then equal to

$$\$961.39 = \$45\left[\sum_{t=1}^{10}\left(\frac{1}{1+(Y/2)}\right)^t\right] + \$1{,}000\left[\left(\frac{1}{1+(Y/2)}\right)^{10}\right]$$

$$= \$45\left[\text{PVIF}_a\left(\frac{Y}{2}\%,10\right)\right] + \$1{,}000\left[\text{PVIF}\left(\frac{Y}{2}\%,10\right)\right]$$

This equation may be solved for Y with either a financial calculator or an iterative solution process such as Newton's Approximation Technique.

METHOD 2

The yield to maturity for this bond may also be solved using bond tables. (A portion of a bond table is presented in Figure A4-5). The first step in using bond tables is to

FIGURE A4-5

Bond Pricing Table (Coupon Rate = 9.0%)

Yield	1 Year	2 Year	3 Year	4 Year	5 Year	6 Year	7 Year	8 Year	9 Year	10 Year
7.00	101.90	103.67	105.33	106.87	108.32	109.66	110.92	112.09	113.19	114.21
7.25	101.66	103.20	104.64	105.98	107.23	108.39	109.48	110.48	111.42	112.30
7.50	101.42	102.74	103.96	105.10	106.16	107.14	108.05	108.90	109.69	110.42
7.75	101.18	102.28	103.29	104.23	105.10	105.91	106.66	107.35	107.99	108.59
8.00	100.94	101.81	102.62	103.37	104.06	104.69	105.28	105.83	106.33	106.80
8.25	100.71	101.36	101.96	102.51	103.02	103.49	103.93	104.33	104.70	105.04
8.50	100.47	100.90	101.30	101.67	102.00	102.31	102.60	102.86	103.10	103.32
8.75	100.23	100.45	100.65	100.83	101.00	101.15	101.29	101.42	101.54	101.64
9.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
9.25	99.77	99.55	99.36	99.18	99.02	98.87	98.73	98.61	98.50	98.39
9.50	99.53	99.11	98.72	98.37	98.05	97.75	97.49	97.24	97.02	96.82
9.75	99.30	98.67	98.09	97.56	97.09	96.65	96.26	95.90	95.57	95.28
10.00	99.07	98.23	97.46	96.77	96.14	95.57	95.05	94.58	94.16	93.77
10.25	98.84	97.79	96.84	95.98	95.20	94.50	93.86	93.29	92.76	92.29
10.50	98.61	97.36	96.22	95.20	94.28	93.45	92.69	92.01	91.40	90.85
10.75	98.38	96.92	95.61	94.43	93.36	92.41	91.54	90.77	90.06	89.43
11.00	98.15	96.49	95.00	93.67	92.46	91.38	90.41	89.54	88.75	88.05
11.25	97.93	96.07	94.40	92.91	91.57	90.37	89.30	88.33	87.47	86.69
11.50	97.70	95.64	93.80	92.16	90.69	89.38	88.20	87.15	86.21	85.37
11.75	97.47	95.22	93.21	91.42	89.82	88.39	87.12	85.98	84.97	84.07
12.00	97.25	94.80	92.62	90.69	88.96	87.42	86.06	84.84	83.76	82.80
12.25	97.03	94.39	92.04	89.96	88.11	86.47	85.01	83.72	82.57	81.55
12.50	96.80	93.97	91.46	89.24	87.27	85.53	83.98	82.61	81.40	80.33
12.75	96.58	93.56	90.89	88.53	86.44	84.60	82.97	81.53	80.26	79.13

refer to the portion of the table that lists yields for the particular coupon rate of interest; in this problem, that would be the 9 percent coupon table. The next step is to convert the current price of the bond to its percentage of par value; this is how bond prices are reported in the financial press. In this example, the price would be reported as 96.14. Scan across the columns until a time to maturity of five years is reached; then scan down the rows until the price closest to 96.14 is found. This corresponds to a yield of 10 percent.

[b] Approximation of Yield. The yield to maturity is the solution to a complex polynomial function. However, there are several formulas used by the industry that are quite simple and provide a close approximation of the true yield. For instance, if an investor plans to hold a bond until it matures, an approximation for the true yield may be calculated as follows:

$$AY = \frac{[F(C) + (P_M - P_P)/(t_M)]}{(P_M + P_P)/(2)}$$

where the subscripts M and P refer to the *maturity price* and *purchase price* of the security, respectively.

Example: A 9 percent coupon bond has five years to maturity. If an investor paid $961.39 for this bond and it will mature for $1,000, what is the approximate yield to maturity for this bond?

$$\text{AY} = \frac{[\$1{,}000(0.09) + (\$1{,}000 - \$961.39)/5]}{(\$1{,}000 + \$961.39)/2}$$

$$= 0.0996457 = 9.96457\%$$

This result is very close to the true yield of 10 percent.

Today, many bonds issued by corporations are callable. This means that the corporation may force the early retirement of the bonds. The usual procedure is to defer the first call date for a period after the initial issue of the bond (e.g., five years). After this period, the bonds may be called by the corporation. If the bond is called, the investor will receive the bond's maturity value plus a call premium, usually a maximum of one year's interest. However, the bondholder then forgoes the right to any other coupons on the bond.

If the bond is selling for more than its maturity value plus one year's interest, there is a very good chance that the bond will be called at the first call date. The yield to call will usually be less than the corresponding yield to maturity. If it is, the financial press will usually report the yield to call as the bond's yield. An approximation for the yield to call may be calculated as follows:

$$\text{CY} = \frac{[F(C) + (P_C - P_P)/t_C]}{(P_C + P_P)/2}$$

where the subscripts C and P refer to the *call price* and the *purchase price*, respectively.

Example: A 9 percent coupon bond is purchased for $1,213.55. The time to maturity for this bond is 20 years, but it is callable in three years at a call price of $1,090. What are the approximate yield to maturity and yield to call?

$$\text{AY} = \frac{[\$1{,}000(0.09) + (\$1{,}000 - \$1{,}213.55)/20]}{(\$1{,}000 + \$1{,}213.55)/2}$$

$$= 0.0717 = 7.17\%$$

$$\text{CY} = \frac{[\$1{,}000(0.09) + (\$1{,}090 - \$1{,}213.55)/3]}{(\$1{,}090 + \$1{,}213.55)/2}$$

$$= 0.0424 = 4.24\%$$

Finally, an investor that sells a bond before maturity may be interested in the actual realized yield on this bond. An approximation for this realized yield may be calculated as

$$\text{RY} = \frac{[F(C) + (P_S - P_P)/t_{IH}]}{(P_S + P_P)/2}$$

Example: A 9 percent coupon bond is purchased for $865. It is sold for $950 after being held for four years. What is the approximate realized yield for this bond?

$$\text{RY} = \frac{[\$1{,}000(0.09) + (950 - \$865)/4]}{(\$950 + \$865)/2}$$
$$= 0.1226 = 12.26\%$$

[c] Government Securities During the Final Coupon Period. The yield on government securities during the final coupon period may be calculated as

$$Y^* = \left[\frac{1 + (C/2)}{(P' + \text{AI}') - 1}\right]\left[\frac{2(t_{\text{CD}})}{t_{\text{PM}}}\right]$$

This is the value that will be quoted as the bond's yield, but it is only an approximation of the true yield. To obtain the true yield, two corrections may have to be made:

1. Since $2(t_{\text{CD}})$ will never be exactly equal to 365 days, the approximate yield must be adjusted by the factor $365/2(t_{\text{CD}})$.
2. It is possible that the bond will not mature on a business day (this is true for most securities). Therefore, the time from purchase to settlement will not be equal to the time from purchase to maturity. If this is the case, the approximate yield must be adjusted by the factor $t_{\text{PM}}/t_{\text{PS}}$.

This implies that the true yield is equal to

$$Y = Y^*\left[\frac{365}{2(t_{\text{CD}})}\right]\left[\frac{t_{\text{PM}}}{t_{\text{PS}}}\right]$$

Example: An investor settles on an 8 percent Treasury bond on September 17, 1992. This bond pays coupons on November 15 and May 15 and matures on November 15, 1992. The maturity value of this bond is $100,000, the current quoted price is $99,960.48, and the amount of accrued interest is $2,717.39. What is the approximate yield and the true yield on this security?

Step 1. There are 184 days between May 15 and November 15. There are 125 days between May 15 and September 17, and 59 days between September 17 and November 15. However, November 15, 1992 is a Sunday. Therefore, the settlement of maturity will not occur until the following day, which is 60 days from the date of purchase.

Step 2. The amount of accrued interest (given in this example) can be calculated as

$$\text{AI} = \$100{,}000\left(\frac{0.08}{2}\right)\left(\frac{125}{184}\right) = \$2{,}717.39$$

The accrued interest per dollar of face value is then equal to

$$\text{AI}' = \frac{\$2{,}717.39}{\$100{,}000} = 0.0271739$$

Step 3. The quoted price, which does not include accrued interest, is $99,960.48. The price per dollar of face value is then equal to

$$P' = \frac{\$99{,}960.48}{\$100{,}000} = 0.9996048$$

Step 4. The approximate yield on this security may now be calculated as follows:

$$Y^* = \left[\frac{1 + (0.08/2)}{(0.9996048 + 0.0271739) - 1}\right]\left[\frac{2(184)}{59}\right]$$
$$= 0.080314343$$

Step 5. The true yield, which is nothing more than a simple interest rate, may then be calculated as

$$Y = 0.080314343\left[\frac{365}{2(184)}\right]\left[\frac{59}{60}\right] = 0.078331947 = 7.8331947\%$$

It can be shown that this value is the true yield, expressed as a simple interest rate, for this security.

Step 1. The interest rate earned over the 60 days from purchase to the settlement of maturity is equal to

$$0.078331947\left(\frac{60}{365}\right) = 0.012876484 = 1.2876484\%$$

Step 2. If a simple interest rate of 1.2876484 percent is earned on an initial investment of

$99,960.48 + $2,717.39 = $102,677.87

after 60 days the investment should have an ending value equal to

$102,677.87(1.012876484) = $104,000

which is exactly equal to the maturity value and the final coupon that will be received on November 16.

[d] Realized Compounded Yield. Some people believe that a *coupon bond's* yield to maturity is the rate of return that they will actually earn (realize) if they hold the bond to maturity. This is usually incorrect. The yield to maturity is nothing more than the bond's internal rate of return and, as such, is a statistical artifact; it is simply the discount rate that when applied to the bond's cash flows will give a present value equal to the price. The only time that it will also be equal to the realized compounded yield (return actually earned) is when the reinvestment rate for the bond's intermediate cash flows is exactly the same as the bond's yield to maturity.

A bond's yield to maturity will understate (or overstate) the realized compounded yield when the true reinvestment rate is greater than (or less than) the calculated yield to maturity. Figure A4-6 illustrates this relationship for a 10 percent coupon bond that pays $50 in interest every 6 months, has 10 years until it matures, and is originally priced to sell at par (that is, its yield to maturity is equal to the coupon rate). If the annual reinvestment rate is also 10 percent (5 percent per 6-month period), the terminal value of the cash flows received plus the interest earned from the reinvestment of those cash flows will be equal to $2,653.30: $1,000 from the maturity value of the bond, $1,000 to be received in the form of coupon payments, and $653.30 from reinvesting the coupons every 6 months to earn a 5 percent, 6-month rate. Given the starting value of $1,000 and the terminal value of $2,653.30, the terminal value ratio is equal to

FIGURE A4-6

Realized Compounded Yields at Different Reinvestment Rates

	Reinvestment Rates		
	8%	10%	12%
Total coupons	$1,000.00	$1,000.00	$1,000.00
Reinvestment income	488.90	653.30	839.28
Terminal value of coupons	$1,488.90	$1,653.30	$1,839.28
Payoff at maturity	1,000.00	1,000.00	1,000.00
Total terminal value	$2,488.90	$2,653.30	$2,839.28
Initial price	$1,000.00	$1,000.00	$1,000.00
Terminal value ratio	2.4889	2.6533	2.8393
Realized compounded yield	9.329%	10.000%	10.713%

Note: Ten percent coupon, 10-year bond, priced at par.

$$\text{TVR} = \frac{\$2{,}653.30}{\$1{,}000} = 2.6533$$

The realized compounded yield can then be calculated as

$$\text{RCY} \Rightarrow 2.65330 = \left[1 + \left(\frac{\text{RCY}}{2}\right)\right]^{20}$$

$$\Rightarrow \quad \text{RCY} = 2(2.65330^{1/20} - 1) = 0.1 = 10\%$$

If the stated annual reinvestment rate is only 8 percent (4 percent per six-month period), the realized compounded yield on an annual basis will be equal to 9.329 percent, and if the stated annual reinvestment rate is 12 percent, the realized compounded yield will be equal to 10.713 percent.

This example shows that it may be unwise for an investor to rely too heavily on the yield to maturity that is reported for a bond. Since reinvestment rates do change over time and may certainly be expected to differ from the yield to maturity, the realized compounded yield may be the only correct measure of the return actually earned by the investor.

[4] Bond Pricing

The quoted price of a bond is equal to the present value of the future cash flows to be received minus the accrued interest. Bonds are priced as of the delivery date. Corporate bonds assume a 180-day coupon period, whereas government securities use the exact number of days within the coupon period. Thus, for bonds not selling at a coupon date, a distinction must be made between corporate and government securities.

[a] Calculation of a Price at a Coupon Date. If a bond is purchased for delivery at a coupon date, accrued interest will be zero. If the bond pays interest on a semiannual basis, all future cash flows to be received will be discounted over full periods at one half of the quoted yield to maturity.

EXAMPLE: A 10 percent coupon bond has a quoted yield of 9.8 percent. If interest on this bond is paid semiannually and the bond has 10 years to maturity, what is its current price?

Step 1. Since this bond is being priced at a coupon date, its accrued interest is zero.

Step 2. This bond will make 20 semiannual payments of \$50, starting 6 months from today, and a final maturity payment of \$1,000.

Step 3. The appropriate discount rate is equal to one half of the quoted yield, or 4.9 percent per six-month period.

Step 4. The current price is equal to the present value of all future cash flows to be received discounted at the six-month rate:

$$P = \$50\left[\sum_{t=1}^{20}\left(\frac{1}{1.049}\right)^t\right] + \$1{,}000\left(\frac{1}{1.049}\right)^{20}$$

$$= \$1{,}012.57$$

[b] Calculation of a Price Between Coupon Dates: Corporate Securities. When a corporate bond is sold between coupon dates, accrued interest is calculated using a 180-day coupon period. The price that an investor must pay is equal to the quoted price plus the accrued interest, which is simply the present value of all future cash flows to be received. This section demonstrates straightforward present value calculations and two additional pricing methods used by the industry.

EXAMPLE: On June 8, 1992, a 10 percent coupon bond with a quoted yield of 6 percent is purchased. This bond makes interest payments on March 15 and September 15 and matures on September 15, 1993. What is the quoted price (add-interest price) and the actual price (flat price) for this bond?

This bond will be delivered for settlement after five business days (one week) or on June 15, 1992. All calculations are made as of the delivery date. The cash flows and timing for this bond may be represented as follows.

Date	*Cash Flow*
June 8, 1992—sale	
June 15, 1992—delivery	
September 15, 1992	\$ 50
March 15, 1993	50
September 15, 1993	1,050

The delivery date of June 15, 1992 falls halfway between the last coupon payment and the next coupon payment (90 days). Accrued interest on this bond may be calculated as

$$\text{AI} = \$1{,}000\left(\frac{0.1}{2}\right)\left(\frac{9}{180}\right) = \$25$$

METHOD 1

To find the present value of the cash flows, the following steps may be used.

Step 1. Calculate the price of the bond as of the next coupon payment date. This price will not include the next coupon to be received:

$$PV = \$50\left[\left(\frac{1}{1.03}\right)^{1}\right] + \$1{,}050\left[\left(\frac{1}{1.03}\right)^{2}\right]$$
$$= \$1{,}038.27$$

Step 2. Add to the calculated price the next coupon to be received:

$$PV = \$1{,}038.27 + \$50$$
$$= \$1{,}088.27$$

This is the present value of all future cash flows to be received, evaluated as of the next coupon date.

Step 3. Discount this new value back to the actual delivery date:

$$PV = \$1{,}088.27\left[\left(\frac{1}{1.03}\right)^{90/180}\right]$$
$$= \$1{,}072.30$$

This is the flat price or the price that the investor actually has to pay.

Step 4. To calculate the quoted price or add-interest price, subtract the accumulated interest:

$$P = \$1{,}072.30 - \$25$$
$$= \$1{,}047.30$$

METHOD 2

The flat price may also be calculated as a one-step process:

$$PV = \$50\left[\left(\frac{1}{1.03}\right)^{90/180}\right] + \$50\left[\left(\frac{1}{1.03}\right)^{270/180}\right]$$
$$+ \$1{,}050\left[\left(\frac{1}{1.03}\right)^{450/180}\right]$$
$$= \$1{,}072.30$$

This is the same price as calculated under Method 1. For this particular problem, there is a slight rounding error when the two industry methods are used.

METHOD 3

The first industry method demonstrated here calculates the quoted or add-interest price. The calculation follows.

Step 1. Using the current yield, calculate the price as of the last coupon payment date. This would be as of March 15, 1992 and would be equal to

$$P_{M83} = \$50[\text{PVIF}_a(3\%, 3)] + \$1{,}000[\text{PVIF}(3\%, 3)]$$
$$= \$1{,}056.57$$

Step 2. Using the current yield, calculate the price as of the next coupon payment date. This would be as of September 15, 1992 and would be equal to

$$P_{S83} = \$50[\text{PVIF}_a(3\%, 2)] + \$1{,}000[\text{PVIF}(3\%, 2)]$$
$$= \$1{,}038.27$$

Step 3. Determine the price appreciation or depreciation over the coupon period:

$$\$1{,}056.57 - \$1{,}038.27 = \$18.30 \text{ of depreciation}$$

Step 4. Determine the proportion of the total price appreciation or depreciation that is applicable to the period from the last coupon date to the delivery date:

$$\$18.30\left(\frac{90}{180}\right) = \$9.15$$

Step 5. The quoted or add-interest price is equal to the price at the last coupon date plus (minus) the applicable price appreciation (depreciation) since the last coupon date.

$$P = \$1{,}056.57 - \$9.15$$
$$= \$1{,}047.42$$

Step 6. The flat price, or actual payment required at delivery, is equal to the quoted price plus accrued interest.

$$\text{PV} = \$1{,}047.42 + \$25.00$$
$$= \$1{,}072.42$$

Method 4

This industry method calculates the flat price of the bond. The steps are as follows:

Step 1. Using the current yield, calculate the price as of the last coupon date. This would be as of March 15, 1992 and would be equal to

$$P_{M83} = \$50[\text{PVIF}_a(3\%, 3)] + \$1{,}000[\text{PVIF}(3\%, 3)]$$
$$= \$1{,}056.57$$

Step 2. Using a 180-day coupon period, find the amount of interest that would result in a 6 percent annual yield if this bond had been bought at the last coupon date and held until the delivery or settlement date.

$$\text{Interest} = \$1{,}056.57\left(\frac{10.06}{2}\right)\left(\frac{90}{180}\right)$$
$$= \$15.85$$

Step 3. The flat price is equal to the price at the last coupon date plus the applicable interest that results in the current yield of 12 percent.

$$\text{PV} = \$1{,}056.57 + \$15.85$$
$$= \$1{,}072.42$$

Step 4. The quoted price is equal to the flat price minus the amount of accrued interest.

$$P = \$1{,}072.42 - \$25.00$$
$$= \$1{,}047.42$$

[c] Calculation of a Price Between Coupon Dates: Government Securities. The price of a government note or bond may also be calculated as the present value of all future cash flows to be received. However, as was true for accrued interest, the exact number of days between coupon payment dates, as well as the exact number of days between the last coupon payment date and the date of delivery, must be determined.

EXAMPLE: A 9⅜ government bond is purchased for delivery on June 22, 1992. The face value of this bond is $100,000 and the current yield is 6 percent, and it pays interest semiannually on May 15 and November 15. If this bond matures on November 15, 1994, what is the quoted price of this bond and what price is actually paid?

Step 1. There are 184 days between May 15 and November 15. There are 38 days between the last coupon date and the date of delivery and 146 days between the date of delivery and the next coupon date.

Step 2. Accrued interest on this bond is equal to

$$\text{AI} = \$100{,}000\left(\frac{0.09375}{2}\right)\left(\frac{38}{184}\right)$$
$$= \$968.07$$

Step 3. The applicable discount rate is one half of the quoted yield, or 3 percent per six-month period. However, the purchaser has to wait 146/184 of a period before receiving its first coupon. The timing and the amount of all cash flows to be received is presented in the following table.

Date of Payment	*Amount*	*Period Until Receipt*
November 15, 1992	$ 4,687.50	146/184
May 15, 1993	4,687.50	1 + 146/184
November 15, 1993	4,687.50	2 + 146/184
May 15, 1994	4,687.50	3 + 146/184
November 15, 1994	104,687.50	4 + 146/184

Step 4. The present value of the cash flows to be received is equal to

$$\text{PV} = \$4{,}687.50\left[\left(\frac{1}{1.03}\right)^{(146/184)}\right] + \$4{,}687.50\left[\left(\frac{1}{1.03}\right)^{(1+146/184)}\right]$$
$$+ \$4{,}687.50\left[\left(\frac{1}{1.03}\right)^{(2+146/184)}\right] + \$4{,}687.50\left[\left(\frac{1}{1.03}\right)^{(3+146/184)}\right]$$
$$+ \$104{,}687.50\left[\left(\frac{1}{1.03}\right)^{(4+146/184)}\right]$$
$$= \$108{,}387.90$$

This is the amount that is actually paid on delivery for this bond.

Step 5. The quoted price is equal to the flat price minus the amount of accrued interest.

P = \$108,387.90 − \$968.07 = \$108,419.83

[d] Government Securities During the Final Coupon Period. Remember from earlier discussion that the yields on government securities during their final coupon period are expressed as a simple interest rate. Given the quoted yield, which may not be equal to the true yield, the actual price of a government security may be calculated as follows:

$$P' = \left[\frac{1 + C/2}{1 + [Y^*(t_{PM})]/[2(t_{CD})]}\right] - AI'$$

EXAMPLE: On June 30, 1992, an investor settles on a 9.75 percent government coupon bond with a quoted yield of 5.25 percent. This bond has a face value of \$100,000 and matures on September 15, 1992. What is the quoted price for this bond and what price is actually paid?

Step 1. There are 184 days from the last coupon payment (March 15) to the date the bond matures. There are 107 days from the last coupon payment to the delivery or settlement date and 77 days from the settlement date to the maturity date.

Step 2. The accrued interest on this bond is equal to

$$AI = \$100{,}000\left(\frac{0.0975}{2}\right)\left(\frac{107}{184}\right)$$
$$= \$2{,}834.92$$

Since the face value of this bond is \$100,000, the accrued interest per dollar of face value is equal to

$$AI' = \frac{\$2{,}834.92}{\$100{,}000} = 0.0283492$$

Step 3. The quoted price on this bond, per dollar of face value, is equal to

$$P' = \left[\frac{1 + (0.0975/2)}{1 + [(0.0525)(77)]/[2(184)]}\right] - 0.0283492$$
$$= 1.0090054$$

which, since the face value of this bond is equal to \$100,000, implies that

P = \$100,000(1.0090054) = \$100,900.54

Step 4. The actual price that the investor is required to pay is equal to

PV = \$100,900.54 + \$2,834.92 = \$103,735

It can be shown that this price results in the quoted yield of 5.25 percent. If the yield is 5.25 percent, this implies that the yield per six-month period is equal to

$$\frac{0.0525}{2} = 0.02625$$

However, for the last coupon period this is the simple interest rate for a 184-day period.

Since the investor holds the bond for only 77 days, the applicable simple interest rate for its holding period is

$$0.02625\left(\frac{77}{184}\right) = 0.010985$$

If the investor purchases the security for a total price of $103,735.46 and earns a simple interest rate of 1.0985 percent, the ending value of its investment should be equal to

$$\$103{,}735.46(1.101985) = \$104{,}875$$

which is exactly equal to the maturity value of this security plus the final coupon to be received.

[5] Premiums and Discounts on Coupon Bonds

If a bond's price is greater than par (yield less than the coupon rate), the bond is selling at a premium. If a bond's price is less than par (yield greater than the coupon rate), the bond is selling at a discount. The calculation of a bond's current price was discussed previously. Once the current price is known, the premium or discount on the bond is simply the difference between this price and the par or maturity value of the bond. However, there is another method that uses the difference between the coupon rate and the quoted yield to determine the amount of premium or discount for the bond.

[a] Calculation of a Discount. The amount of discount for a bond is equal to the present value of an annuity stream that represents the difference between the coupon rate and the quoted yield.

EXAMPLE: On April 1, 1992, an investor takes delivery of a 5 percent coupon bond. This bond has a face or par value of $1,000 and pays interest on April 1 and October 1. This bond matures on October 1, 1996, and the current yield to maturity is 6 percent. What is the amount of discount for this bond?

METHOD 1

Calculate the current price for this bond. It is equal to

$$\begin{aligned} P &= \$25[\text{PVIF}_a(3\%, 9)] + \$1{,}000[\text{PVIF}(3\%, 9)] \\ &= \$961.07 \end{aligned}$$

The discount for this bond is then equal to

$$\begin{aligned} \text{Discount} &= \$1{,}000.00 - \$961.07 \\ &= \$38.93 \end{aligned}$$

METHOD 2

Any discount on this bond will be due to the difference between the coupon rate of 5 percent and the current yield of 6 percent. For this security to sell at par with a yield of 6 percent, it would have to pay interest of $30 every six months. Since it actually pays only $25 every six months, this is a difference of $5. This difference of $5, which will occur at